THE BANACH TARSKI PARADOX

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Introduction

There are things that seem incredible to most men who have not studied Mathematics ∼ Aristotle

Mathematics, in its earliest form, was an array of methods used to quantify, model, and make sense of the world around us. However, as the study of this ancient subject has advanced, it has morphed into its own self contained universe, which while touching our own profoundly and often beautifully, is a fundamentally different entity, created from the abstractions of our minds. As Albert Einstein, in his '*Sidelights on Relativity*', so elegantly put it : "*As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality*". This was written in 1922, two years before the publication of [BAN24], which provided a proof of the Banach-Tarski Paradox, one of the most striking exemplifications of this quote .

The Banach Tarski Paradox. *It is possible to partition the unit ball in $\mathbb{R}^3$ into a finite number of pieces, and rearrange them by isometry (rotation and translation) to form two unit balls identical to the first. More generally, given any two bounded subsets of $\mathbb{R}^3$ with non-empty interior* [1]*, we may decompose one into a finite number of pieces, which can then be rearranged under isometry to form the other.*

It is clear that mathematical truths do not apply to us ontologically, from this claim, if nothing else. What then, do the truths of this vast subject refer to? Modern mathematics is constructed from sets of axioms, statements which we take to be self evident, and which then qualify every further step we take. We naturally choose such axioms as seem patently obvious to us in the real word, in order that the relationships we then make have relevance outside of the abstract realms they inhabit. Deriving results based on the premise that statements can be true and false would be meaningless, for instance. However, mathematical arguments rely heavily on concepts that we cannot form an empirically derived logical basis for, notably that of infinity. It is therefore necessary to build a set of rules on which arguments incorporating such concepts can be based, that balance consistency (arguments

[1] As defined in [PRE08]

built on these rules cannot give contradictory results), with sufficient richness to afford a level of profundity to the results thereby obtained. Historically, no such axiomatic system was explicitly constructed, and any such logical operations as seemed legitimate in the real world were extrapolated to mathematics. Being able to define objects with arbitrary parameters, and construct sets of such objects, was therefore deemed permissible. After all, defining a set as 'the collection of all objects that satisfy condition X', seems to pose no obvious contradictions. However, developments at the beginning of the 20^{th} century forced mathematicians to take a closer look:

Russell's Paradox. *Let* $R = \{x : x \notin x\}$*. Then* $R \in R \Leftrightarrow R \notin R$

Proof. R is the set of all sets that are not members of themselves. If R is a member of itself, then R is not in the set of sets that are not members of themselves, so R is not in R. If it is not a member of itself, then it is a member of R. □

Since the existence of this set, imagined by the philosopher Bertrand Russell, contradicts a basic 'law of nature' fundamental to our understanding of logic: that of a statement being either true or false, , we label the construction a 'paradox' and do not allow its construction in any form of mathematics where we want truth and falsehood to be disjoint entities. Unfortunately, this implies that we must restrict the allowable properties and operations applicable on sets, and form such an axiomatic system as I previously proposed.

Russell's Paradox led to a rush of activity in the emerging fields of Set and Model theory, as mathematicians sought for consistency, and a set of rules that would allow no such paradoxes to occur. Indeed Hilbert's Second Problem, one of a collection of the most important unsolved questions in Mathematics at the turn of the 20th century, was to prove that the axioms of arithmetic are consistent. Gödel, unfortunately, in his Second Incompleteness Theorem, proved that no axiomatic model can be proved to be consistent within its own logic. Indeed, through his First Incompleteness Theorem, he proved that the axioms cannot be both consistent and complete [GOD31]. Some logicians dispute the validity of applying these theorems to Hilbert's problem. A discussion is outside the scope of this essay. The reader could refer to eg [DET90] for more information. Suffice to say that, currently, the most widely accepted axiomatic system for non-constructive mathematics is known as the Zermelo Fraenkel Model, with the axiom of choice (ZFC). ZFC can be thought of a set of 'rules' within which we can define the structures we create, in order that consistency is achieved. It negates Russell's Paradox through inclusion of the "*Axiom Schema of Separation*" [HAL74], which disallows existence of R. Unfortunately, we know, from Gödel's aforementioned Second Incompleteness Theorem, that it is impossible to prove the consistency of ZFC using only arguments derived from ZFC. Deeper study in this area ends up questioning the foundations of the notion of mathematical proof and rigour itself. Note that the axiom of choice is not part of ZF, due to its traditionally controversial role in mathematics. It plays a sufficiently important role in the Banach Tarski Paradox that an explicit exposition is necessary:

The Axiom of Choice. *For any collection (possibly infinite)*[2] *of non-empty sets; there exists a choice function; that is, a function that takes precisely one element from each of the sets; defined on the collection*

Upon observation, Russell's paradox is qualitatively different from Banach Tarski. It is an argument, derived from a set of axioms of logic, that contradicts consistency of the elements of this set: logic decries that a statement can be both true and false. Banach Tarski, on the other hand, is perfectly consistent within ZFC, so the term 'paradox' is, in some senses, a misnomer. It merely counters our intuition that the collective volume of a disjoint collection of objects is constant when subjected to isometries. However, as such, it is consistently cited as a reason to disallow the axiom of choice in mathematics. Yet this is an empirical intuition derived from the physical world, rather than analytically from the mathematical universe, else, there would be some inconsistency in the application of the paradox to ZFC. What if we were to base all mathematics on such physical intuitions? For one, we would have to disallow the use of infinitely small sets, citing the inherent granularity of matter, and the absurdity of being able to splice an atom into a countable collection of subsets, which could then be rearranged such that they were dense in the universe. I will leave it for the reader to decide if such a decision would be valid. The purpose of this essay is to serve as a reminder to the reader that mathematics operates within its own purview, distinct from our own, and that concepts we deem innate may not be haphazardly assumed mathematically.

Background

This section, while not necessary for the proof of Banach Tarski that follows, provides an introduction to the techniques and concepts that underly the proof. It is hoped that the reader will consider the possibility of a rigorous proof of the paradox as being slightly less unbelievable after appraisal of this section.

The notion of infinity, while ubiquitous in mathematics, is fundamentally an abstract concept far removed from actuality. Intuition, therefore, is not a relevant tool in our exploration of this concept. The following construction exemplifies this:

Hilbert's Hotel.
David Hilbert runs a hotel with a (countably) infinite number of rooms, of which a finite number are unoccupied. Hilbert, wishing to maximise profits, wants every room to be filled.

[2]Note that the finite case does not require the axiom of choice, it can be proved by induction

In fact he is thinking of building a countably infinite extension. However, no new guests arrive. Can we help?

Of course. Suppose there are n empty rooms. We label them 1 to n. Having stayed up all night numbering each occupied room, from $n+1$ onwards, (possible as there are only countably infinite rooms), we merely request every guest to move to the room with number equal to their current one minus n. Suppose Hilbert builds an infinite extension, numbering the rooms $B1, B2, B3$.... Then, we instruct all guests in the original building with a prime numbered room to move into the extension, with the guest in the room numbered with the n^{th} prime number to move into BN. Since there are, infinite primes, the extension is filled. To reoccupy the prime numbered rooms, we instruct every guest with a room number of the form p^m, where $m \in \mathbb{N}$, and p is Prime, to move to the room numbered p^{m-1}.

This odd construction relies on the fact that the statements the cardinality of all sets of countably infinite order is identical, that is, we can map one to another bijectively. While possibly counterintuitive to the layperson, very few who have studied mathematics, and understood concepts such as countability of the rationals, would have trouble accepting this fact in their intuition of the infinite.

Hilbert's Hotel on the Unit Sphere.
A more relevant formulation, in terms of the Banach Tarski paradox, is to equate the previous example to a similar construction on the unit sphere: S^1. Take a point $x \in S^1$ and let x_n be the point obtained by rotation of x clockwise by n radians. Consider $X = \{x_n\}_{n=1}^{\infty}$, which consists of distinct points (no two elements are the same as 2π, the identity rotation in radians, will not divide any natural number). Identifying the room numbered n in Hilbert's hotel with x_n; we can rotate X anticlockwise by m radians, and the image of this rotation will be $X \cup \{x_{-i}\}_{i=1}^{m}$ Thus the image of the rotation is a strict superset, containing m extra points. This example is proved more rigorously later on, see Theorem 3.11. Its main purpose here is to show that the image of a set under rotation can strictly contain the original set, one of the major counterintuitive elements of Banach Tarski. Note that an infinite extension, ie a countable extra number of points in the image, is much harder to obtain. Rotation by n radians is equivalent to shifting each guest n rooms. To accomodate countably infinite guests, different guests will have to move a different number of rooms. Also note that the axiom of choice was not required here.

Non-measurable sets.
We have just shown that points can be added to a set by rotating it. However, to change the Lebesgue measure of a set by isometry, an uncountable number of points must be added. Clearly, a simple bijection such as $f(x) = 2x$ will alter the length of a line, but involves translation of every point on the interval, an uncountable number of isometries. We aim

to achieve this in a countable such number. Since the union of a countable number of disjoint sets is required to have the same measure as their sum (by definition of measure), this would be a more interesting achievement, measure theoretically. The crux of Banach Tarski, is that finite isometries of a set can alter its Lebesgue measure. I will briefly show how countably many such isometries, applied to a non measurable set, can result in either a bounded set, or one covering the reals:

Vitali Sets. *$V \subset \mathbb{R}$ is a Vitali Set if, for each $r \in \mathbb{R}$, it contains exactly one element v such that $v - r \in \mathbb{Q}$. In other words, it is an image of a choice function on the set of cosets comprising the quotient group $\mathbb{R}/\mathbb{Q}$.*

Note how the existence of this image depends on the axiom of choice. Also, since each coset of $\mathbb{R}/\mathbb{Q}$ is a shifted copy of $\mathbb{Q}$, and is thereby dense in $\mathbb{R}$, we may pick each representative element in an arbitrarily small interval. Therefore, we can pick a Vitali Set contained within the same interval.

Claim. Let V be a Vitali Set. For any $q_1, q_2 \in \mathbb{Q}$; $\{q_1 + V\} \cap \{q_2 + V\} = \emptyset$*:*

Proof. Suppose there was an element in the intersection. Then $\exists v_1, v_2 \in V : q_1 + v_1 = q_2 + v_2$. But then v_1 can be obtained from v_2 by addition of a rational number, implying they are in the same coset of $\mathbb{R}/\mathbb{Q}$. This contradicts construction of the Vitali Set, as all cosets are disjoint, and V must contain exactly one member from each of them.

□

Claim. *For any $\epsilon > 0$, we can construct a Vitali Set V, together with countable sequences: $\{a_i\}_{i=1}^{\infty}, \{b_i\}_{i=1}^{\infty} \subseteq \mathbb{Q}$ such that $\displaystyle\bigcup_{i=1}^{\infty} a_i + V \subseteq (0, \epsilon)$ and $\displaystyle\bigcup_{i=1}^{\infty} b_i + V = \mathbb{R}$, such that elements within each countable union are pairwise disjoint.*

Proof. We have already shown that a Vitali Set can be constructed in any open interval, so let $V \subseteq (\frac{1}{4}\epsilon, \frac{3}{4}\epsilon)$ be such a set. There are countably many rationals in any interval, so enumerate the rationals in the interval $(0, \frac{1}{8}\epsilon)$ as $\{a_i\}_{i=1}^{\infty}$; and $\mathbb{Q}$ as $\{b_i\}_{i=1}^{\infty}$. Obviously $\bigcup_{i=1}^{\infty} a_i + V \subseteq (0, \epsilon)$. To prove the other part of the claim, take any $r \in \mathbb{R}$. It must be an element of a coset of $\mathbb{R}/\mathbb{Q}$. Therefore, there is a $v \in V$ in the same coset, and, for some $b_k \in \mathbb{Q}$; $r = b_k + v$, and r is therefore covered in the union $\bigcup_{i=1}^{\infty} b_i + V$ □

In conclusion, we have essentially taken a countable number of disjoint translations of a set, which collectively cover $\mathbb{R}$, and reshuffled it into a different sequence of disjoint translations that fit inside an interval!

The following chapters constitute a proof a the Banach Tarski Paradox. While many of the finer details of the proof, may be unique to this essay, the fundamental reasoning is derived from [WAG85], and as such, I do not claim originality for the exposition.

1. Paradoxical Sets

Definition 1.1. *Paradoxical and Countably Paradoxical Action of a Group on a Set*

Let X be a set, and G be a group with a left action defined on X. Let $E \subseteq X$ be a set with pairwise disjoint subsets $A_1...A_n$; $B_1...B_m$ and corresponding elements $g_1...g_n$ and $h_1...h_m$ such that

$$E = \bigcup_{i=1}^{n} g_i(A_i) = \bigcup_{i=1}^{m} h_i(B_i)$$

Then we say E is paradoxical with respect to G. If the disjoint subsets and their corresponding elements are countable rather than finite, then E is countably paradoxical with respect to G.

Example 1.2. *Banach Tarski Paradox*
Every Ball in $\mathbb{R}^3$ *is paradoxical with respect to the group of isometries on* $\mathbb{R}^3$

This alternative formulation of the Banach-Tarski Paradox places it firmly within the realm of the theory of paradoxical sets. In the rest of the section, we will gain some tools for the eventual proof of the paradox by improving our ability to classify sets (and indeed groups) that are paradoxical with respect to a group.

Definition 1.3. *Paradoxical Groups*

If a group is paradoxical with respect to itself under the natural group action $g \circ x = gx$ where $g, x \in G$, then we define it to be a paradoxical group. Note that such a group cannot have finite order (unless it is the trivial group), as the cardinality of the two disjoint subsets, which are acted on to form the whole group, is necessarily greater or equal to that of the whole group, as a one-to-many group action is not allowable.

We take a small detour in order to define free groups, which play a critical role in subsequent chapters

Definition 1.4. *Reduced words*

For any group G, we can pick an arbitrary generating set, and form a notation such that each generator is represented by a 'letter', and its inverse by the 'inverse letter' (ie writing the inverse of 'a' as 'a^{-1}') . We name the collection of such letters and their inverses an 'alphabet', and note that any member of the group, being a composition of generating elements, may be represented as a sequence of letters, where adjacency of elements of the sequence denotes composition under the group action. This representation is not unique. For instance, we may trivially insert a letter followed by its inverse on the end of a word, without altering the group element it represents, as these two compose to give the identity element. To obtain a reduced word from a general word, we delete all such adjacent pairs of letters and their inverses. Thus we equate, for example the words ' gh' and ' $gg^{-1}gh$'. The identity element is the empty word. Group elements are not necessarily uniquely represented by reduced words either. For example, in a finitely generated abelian group, any permutation of a set of letters will give the same word.

Definition 1.5. *Free Groups*

Suppose we construct an alphabet of n letters (with no mutually inverse elements), and their inverses , and form a group consisting of reduced words in this alphabet. This is known as the free group on n generators. While for an arbitrary group, we have noted above that multiple words may represent the same element, we see here by construction of the group that any two distinct words will not represent the same element. This makes intuitive the usage of the term 'free', as we see that there is no relationship between distinct letters: one letter may not be represented as a combination of other letters without sacrificing the condition that there is a one to one correspondence between words and group elements. Note also that every element has infinite order: if it had order n; then its word could be repeated n times to yield a nontrivial word equivalent to the identity.

Example 1.6. *F_2; The Free Group on Two Generators*

Let us call the generating set $S = \{a, b\}$; then every element of F_2 is represented as a word, with letters from the set $\{a, b, a^{-1}, b^{-1}\}$. The cardinality of the group is countably infinite, as it consists of a countable number of lengths of word, each with a finite number of words. This group is central to proof of the Banach Tarski Paradox.

Lemma 1.7. *Let G be a group with elements $\{a, b\}$. If the subgroup of words generated by $\{a, b\}$ is such that no nonempty word 'ending' in b represents the identity element, then this subgroup is isomorphic to F_2*

Proof. This assertion is equivalent to stating that no nonempty word generated by $\{a, b\}$ represents the identity element, as for any word p; $p = Id \Leftrightarrow b^{-1}pb = Id$. Suppose we have distinct reduced words w_1, w_2 generated by S, and representing the same element. Then $w_1 {w_2}^{-1}$ is a nontrivial word representing the identity. So all reduced words are unique, and the subgroup generated by S is isomorphic to the free group on two generators. This lemma is useful in later chapters. □

Definition 1.8. *Free Semigroups*

A semigroup is a set satisfying every group axiom save the existence of an inverse. When a semigroup is a subset of a larger semigroup, it is referred to as a subsemigroup. Free semigroups are constructed identically to free groups, except the alphabet on which the words representing elements of the group are constructed contains no inverse elements. So, for instance, the free semigroup on two generators is the set of words from the alphabet $\{a, b\}$, such that every different word represents a different element.

Theorem 1.9. *The Free Group on 2 generators is paradoxical*

Proof. Let F_2 be generated by: $\{a, b\}$. Every element of the group, when described as a reduced word, 'begins' with a member of the alphabet $A = \{a, a^{-1}, b, b^{-1}\}$. Let the function $W : A \to \mathcal{P}(F_2)$ (*the power set of* F_2) organise the group into equivalence classes determined by the starting letter of the element of the group. So, for example, $W(a)$ is the set of elements of F_2 whose alphabetic representation begins with a. Note that $W(a), W(a^{-1}), W(b)$, and $W(b^{-1})$ are pairwise disjoint, and partition F_2.

Take $g \notin W(a)$. Obviously

$$a^{-1}g \in W(a^{-1})$$

Therefore;

$$a(a^{-1}g) \in a(W(a^{-1})$$

But aa^{-1} is the identity element, so

$$g \in aW(a^{-1})$$

Therefore,

$$F_2 = W(a) \cup aW(a^{-1}) = W(b) \cup bW(b^{-1})$$

.

Moreover, both decompositions of F_2 are pairwise disjoint.

So F_2 is a paradoxical group

□

Theorem 1.10. *Any Group containing the the Free Semigroup has a non-empty paradoxical subset*

Recall that a subsemigroup is a subset of a group containing the identity element and having the property of closure under the group operation. Our claim is that any group G containing a subsemigroup isomorphic to the Free Semigroup on two generators, has a non-empty paradoxical subset.

Proof. Suppose $S \subset G$ is isomorphic to the free semigroup on two generators. Then, for some $a, b \in G$, S consists of sequences of these letters. Let $W(a)$ and $W(b)$ be the sets of sequences with a and b as the first elements, Note that these are disjoint subsets of S. Furthermore,

$$S = a^{-1}W(a) = b^{-1}W(b)$$

Hence S is paradoxical with respect to G □

Theorem 1.11. *If, for any set X, there exists a group action that is faithful on X with respect to a paradoxical group G, then X is paradoxical with respect to G (AC)*

Recall that the action of a group is faithful if there are no trivial fixed points, ie, $g \circ x = x \Rightarrow g = Id \; \forall \; x \in X$

Proof. Let $\{A\}_{i=1}^{n}$, $\{B\}_{j=1}^{m}$ and $\{g\}_{i=1}^{n}$, $\{h\}_{j=1}^{m}$ be respectively the disjoint subsets and group elements that show G to be paradoxical.

Let $Gx = \{g \circ x; g \in G\}$; the orbit of $x \in X$. We know from elementary Algebra, that X can be partitioned into orbits, ie every element is in some orbit, and for any $x, y \in X$, either $Gx = Gy$ or $Gx \cap Gy = \emptyset$

Using the Axiom of Choice, we can pick a set M which contains exactly one element of each orbit of X.

Claim : $\{gM : g \in G\}$ forms a pairwise disjoint partition of X.

To prove the claim let us first prove that this family covers X. Pick any $y \in X$
$y \in Gx$ for some $x \in X$
By construction of M, $\exists m \in M : Gm = Gx$. Hence, $y = h \circ m$ for some $h \in G$
Next, let us show that this family of sets is pairwise disjoint:
Suppose $g \circ m_1 = h \circ m_2$ where $g \neq h$, $g, h \in G$; $m_1, (\neq) m_2 \in M$

Then $h^{-1}g \circ m_1 = m_2$

$$\Rightarrow Gm_1 = Gm_2$$

$$\Rightarrow m_1 = m_2 \text{ by construction of } M$$

But this implies that $h^{-1}g$ is a fixed point of M, which is a contradiction as the group action has no nontrivial fixed points by assumption.

Let $\{A_i^*\} = \{\bigcup g(M) : g \in A_i\}$ and $\{B_i^*\} = \{\bigcup g(M) : g \in B_i\}$

Then $\{A_i^*\}_{i=1}^{n}, \{B_i^*\}_{j=1}^{n}$ are all pairwise disjoint. Furthermore,

$$G = \bigcup_{i=1}^{n} g_i(A_i) \Rightarrow X = \bigcup_{i=1}^{n} g_i(A_i^*)$$

as gM is a member of the union for every $g \in G$. Similarly

$$G = \bigcup_{i=1}^{m} h_i(B_i) \Rightarrow X = \bigcup_{i=1}^{m} h_i(B_i^*)$$

As claimed, X is G-paradoxical □

Corollary 1.12. *Any group with a paradoxical subgroup is itself paradoxical (AC)*

Proof. The natural action of any subgroup on a group is faithful, as

$$g \circ x = x \Rightarrow g = Id$$

□

Theorem 1.13. *If, for some group G, there exists a set X such that X is G-Paradoxical, then G is paradoxical itself*

Proof. Let us first pick an $x \in X$, and construct a paradoxical decomposition of the orbit Gx.

We know, for some $g_1...g_n; h_1...h_m \in G$, and pairwise disjoint $A_1^*...A_n^*; B_1^*...B_m^* \in X$, that

$$X = \bigcup_{i=1}^{n} g_i(A_i^*) = \bigcup_{i=1}^{m} h_i(B_i^*)$$

Let $A_i = \{g : g(x) \in A_i^* \cap Gx\}$; and $B_i = \{g : g(x) \in B_i^* \cap Gx\}$

Then $$\bigcup_{i=1}^{n} g_i(A_i) = \bigcup_{i=1}^{n} \{g' : g'(x) \in g_i(Gx) \cap g_i(A_i^*)\}$$

But $g_i(Gx) = Gx \; \forall g_i \in G$ by definition of orbit, and

$$\bigcup_{i=1}^{n} g_i(A_i) = X$$

.

So we have

$$\bigcup_{i=1}^{n} g_i(A_i) = \{g' : g'(x) \in (Gx)\}$$

Clearly this includes the whole group G. So we have:

$$\bigcup_{i=1}^{n} g_i(A_i) = G$$

Using the same reasoning, we can also obtain

$$\bigcup_{i=1}^{n} h_i(B_i) = G$$

So we have arrived at a paradoxical decomposition of G

□

Von Neumann Conjecture. *The only paradoxical groups are those which have a subgroup isomorphic to the free group on two generators.*

This conjecture is false. It was originally formulated in the context of amenable groups, a class of groups for which one can attach a finite measure that is left-invariant ($\mu(A) = \mu(gA)$ $\forall g \in G$) [3]. Von Neumann speculated that the amenability of a group was equivalent to it not containing the free group on 2 generators as a subgroup. Tarski's theorem [WAG85] proved that a group's amenability is equivalent to its non-paradoxicity. Thus we arrive at the statement heading this paragraph. The conjecture was disproven in 1980, by Alexander Ol'shanskii [WAG85], using so-called "Tarski Monster Groups", which have members that are both paradoxical and lacking a subgroup isomorphic to the free group on two generators.

2. The Hausdorff Paradox

Let S^2 be the unit sphere in $\mathbb{R}^3$ and SO_3 be the group of rotations on the unit sphere. The statement of the Hausdorff Paradox is that there exists a countable subset $D \subset S^2$ such that $S^2 \backslash D$ is paradoxical with respect to SO_3. The proof of this works in several stages. We first construct a subgroup of SO_3 isomorphic to the free group on two generators. Corollary 1.12 thereby establishes that SO_3 is itself paradoxical. We then find a maximal subset of S^2 on which F_2 has a faithful action, which turns about to be S^2 with a countable number of points removed. The paradox is then a consequence of Theorem 1.11.

Lemma 2.1. *SO_3 contains a subgroup F_2 isomorphic to the Free Group on two generators*

Proof. Let $a^{\pm 1}$ be the clockwise (anticlockwise) rotations of angle $cos^{-1}(\frac{3}{5})$ around the x-axis, and $b^{\pm 1}$ be the rotations of the same angle around the z-axis. These rotations can be represented in matrix form as follows:

$$a^{\pm 1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & \mp 4/5 \\ 0 & \pm 4/5 & 3/5 \end{pmatrix} \qquad b^{\pm 1} = \begin{pmatrix} 3/5 & \mp 4/5 & 0 \\ \pm 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We can see that these four rotations generate a subgroup of SO_3. We want to show that it is isomorphic to the free group on two generators. By Lemma 1.7, it is sufficient to show that no 'words' ending in 'b' can represent the identity rotation:

[3] A rigorous formulation and detailed discussion of amenable groups can be found in [GRE69], but a simple definition will suffice for our purposes

Let W be the set of words ending in b, and $W_c \subset W$ be the class of such words with length c. So $\bigcup_{c=1}^{\infty} W_c = W$. We claim that $\forall w \in W_c$, $w(1,0,0)$ is of the form $\frac{1}{5^c}(i,j,k)$, where $i,j,k \in \mathbb{Z}$ and $5 \nmid j$. This implies no such w can be equivalent to the identity rotation, as $Id(1,0,0) = \dfrac{1}{5^c}(5^c,0,0)$ and $5 \mid 0$

The first part of our claim, that is $i,j,k \in \mathbb{Z}$, is proven using induction on word length. If w has length one, then $w = b^{\pm 1}$, in which case $w(1,0,0) = \frac{1}{5}(3, \pm 4, 0)$, which agrees with our claim.

Assume $\{i,j,k\} \in \mathbb{Z}$ for all w' of length $c-1$. Then, for all w of length c, $w = a^{\pm 1}w'$ or $w = b^{\pm 1}w'$ where w' is some word of length $c-1$.

$w'(1,0,0) = \frac{1}{5^{c-1}}(i',j',k')$ for some $\{i',j',k'\} \in \mathbb{Z}$

$$w = a^{\pm 1}w' \Rightarrow w(1,0,0) = \frac{1}{5^c}(5i, 3j \mp 4k, \pm 4j + 3k) \in \frac{1}{5^c}\mathbb{Z}^3$$

$$w = b^{\pm 1}w' \Rightarrow w(1,0,0) = \frac{1}{5^c}(3i + 4j, \pm 4i + 3j, 5k) \in \frac{1}{5^c}\mathbb{Z}^3$$

So, by induction, a word w of length c rotates $(1,0,0)$ to produce a vector of the form $\frac{1}{5^c}\mathbb{Z}^3$

It remains to show that the second element of the vector produced by such a rotation is not divisible by 5^{c+1}

We have shown that this is true for a word of length one (ie $\pm b$). Again, we prove by induction. Consider a word w of length c such that $c \geq 2$. Label the first two letters of w: q and r. Let $\rho = q^{-1}w$ and $\sigma = q^{-1}r^{-1}w$. By the proof of the previous part of the claim, we can say

$$w(1,0,0) = \frac{1}{5^c}(i,j,k) \qquad \rho(1,0,0) = \frac{1}{5^{c-1}}(i',j',k') \qquad \sigma(1,0,0) = \frac{1}{5^{c-2}}(i'',j'',k'')$$

For the purposes of the inductive step, we can assume that j' is not divisible by 5. For any w, one of the following four statements must hold true, depending on the first two letters of its representation.

$$w = (a^{\pm 1})(a^{\pm 1})\sigma \tag{1}$$

$$w = (a^{\pm 1})(b^{\pm 1})\sigma \tag{2}$$

$$w = (b^{\pm 1})(a^{\pm 1})\sigma \tag{3}$$

(4) $$w = (b^{\pm1})(b^{\pm1})\sigma$$

Using simple matrix multiplication, we can derive j in terms of $\{i', j', k', i'', j'', k''\}$ and thereby show that it is not divisible by 5:

(1) $$j' = 3j'' \mp 4k'' \qquad k' = \pm 4j'' + 3k''$$

$$\begin{aligned} j = 3j' \mp 4k' &= 3j' \mp (\pm 16j'' + 12k'') &&= 3j' - 16j'' \mp 12k'' \\ &= 3j' + 3(3j' \mp 4k') - 25j'' &&= 6j' - 25j'' \end{aligned}$$

By assumption, $5 \nmid 6j'$; and since $5 \mid 25j''$ we ascertain that $5 \nmid j$

(2) $$j' = 3j'' \pm 4i'' \qquad k' = 5k''$$

$$j = 3j' \mp 4k' = 3j' \mp 20k''$$

By assumption, $5 \nmid 3j'$; and since $5 \mid 20k''$ we ascertain that $5 \nmid j$

(3) $$i' = 5i''$$

$$j = 3j' \pm 4i = 3j' \pm 20i''$$

By assumption, $5 \nmid 3j'$; and since $5 \mid 20j''$ we ascertain that $5 \nmid j$

(4) $$j' = 3j'' \mp 4i'' \qquad i' = \pm 4j'' + 3i''$$

$$\begin{aligned} j = 3j' \mp 4i' &= 3j' \mp (\pm 16j'' + 12i'') &&= 3j' - 16j'' \mp 12i'' \\ &= 3j' + 3(3j' \mp 4i') - 25j'' &&= 6j' - 25j'' \end{aligned}$$

We arrive at the same result as in case one, and can similarly conclude that $5 \nmid j$

We now know that $w(1,00) \neq (1,0,0)$, where $w \in W$. We have already established that the truth of this property in W implies its truth for all words in the subgroup. So the subgroup generated by the rotations a and b is isomorphic to the Free Group on two generators.

□

Corollary 2.2. *The group of isometries in $\mathbb{R}^n$, where $n \geq 3$, is paradoxical*

Proof. First let us prove the case $n = 3$: We have shown that a group of rotations in 3 dimensions is paradoxical. Since this is a subgroup of the group of isometries in $\mathbb{R}^3$, Corollary 1.12 ensures that the claim is true for $n = 3$. Any group of isometries on a higher dimensional space contains those of 3-space as a subgroup. Therefore, Corollary 1.12 again ensures they are paradoxical. □

Theorem 2.3. *There is a countable set D such that $S^2 \backslash D$ is paradoxical with respect to F_2, the previously constructed subgroup of SO_3*

Proof. Recall Theorem 1.11. It remains to show that there is a countable set D such that F_2 has a faithful group action on $S^2 \backslash D$. Each element of $f \in F_2$, being a rotation, has two fixed points. There are countably many such elements f (see Example 1.6). Therefore, there are at most countably many fixed points in the action of F_2 on S^2. Letting D be this set, we can see that F_2 has a faithful group action on $S^2 \backslash D$

□

We have arrived at a result not far removed from Banach Tarski, having shown that S^2 with only a countable set of points removed is paradoxical under the group of rotations (and therefore isometries). This is extremely counterintuitive in its own right. However, using only the theory we have built up about paradoxical sets, this is the closest we can come to proving Banach-Tarski. We can only 'transport' the paradoxicity of a group to a set on which it acts by showing there are no fixed points in the action (Theorem 1.11). This is impossible where the group acts on the unit ball by rotating it, as any rotation preserves uncountably many fixed points in $\mathbb{R}^3$. To bypass this seemingly insurmountable problem, we resort to building a new machinery; that of equidecomposable sets.

3. Equidecomposability

Definition 3.1. *Let a group G act on a set X, with subsets A, B. If there exists $g \in G$ such that $g(A) = B$, then A and B are G-congruent.*

Definition 3.2. *Let a group G act on a set X. $A, B \subseteq X$ are finitely (countably) G-equidecomposable if we can decompose them such that:*

$$A = \bigcup_{i=1}^{n} A_i \qquad B = \bigcup_{i=1}^{n} B_i$$

where n is finite, $\{A_i\}, \{B_i\}$ are both families of pairwise disjoint sets, and $\exists g_1 ... g_n \in G$ such that $g_i(A_i) = B_i$ We write $A \sim_G B$

Corollary 3.3. *A set X is paradoxical with respect to G iff X contains disjoint subsets A, and B, such that $A \sim_G X$ and $B \sim_G X$*

Theorem 3.4. *G-Equidecomposability is an equivalence relation*

Proof. 1. $A \sim_G A$ under the action of Id_G

2. $A \sim_G B \Rightarrow B \sim_G A$ as $g_i(A_i) = B_i \Leftrightarrow g_i^{-1}(B_i) = A_i$

3. Suppose $A \sim_G B$ with n pieces; and $B \sim_G C$ with m pieces. So, for some partitions of A, B, and C, and families $\{g_i\}; \{f_i\}$ of elements of G we have :

$$\bigcup_{i=1}^{n} g_i(A_i) = \bigcup_{i=1}^{n} B_i \qquad \text{and} \qquad \bigcup_{i=1}^{m} f_i(B_i^*) = \bigcup_{i=1}^{m} C_i$$

Where $\{A_i\}$ is the partition of A; $\{B_i\}$ and $\{B_i^*\}$ are unrelated partitions of B; and $\{C_i\}$ partitions C. Take

$$S_{ij} = A_i \cap g_j^{-1} f_j^{-1}(C_j)$$
$$h_{ij} = f_j g_i$$

Then: $$\bigcup_{i=1}^{n} \bigcup_{j=1}^{m} h_{ij}(S_{ij}) = C$$

And since $\{S_{ij}\}$ is a family of disjoint (not necessarily non-empty) subsets of A, we have $A \sim_G C$

□

Corollary 3.5. *If a set X is paradoxical with respect to some group G, then so too is the equivalence class of sets that are G-equidecomposable with X*

Proof. X is paradoxical $\Leftrightarrow$ There exist pairwise disjoint, nonempty $A, B \subset X$ such that $A \sim_G X$ and $B \sim_G X$. Suppose $X' \sim_G X$. We take A', B' as the images of A, B under the equidecomposability map from X to X'; whose bijectivity ensures that they are disjoint. A' and B' are equidecomposable with A, and B respectively, under the equidecomposability map, and, by transitivity, with X' Therefore they form a paradoxical decomposition of X' □

Since we have now obtained an equivalence relation on sets with respect to equidecomposability, it is possible to extend this to a partial ordering.

Definition 3.6. *Let a group G act on a set X, with subsets A, B, and an equidecomposability relation $\sim$. If A is equidecomposable with a (not necessarily proper) subset of B, we write: $A \preceq B$*

Lemma 3.7. *$\preceq$ is a partial ordering*

Proof.
Reflexivity: $A \sim A \subseteq A \Rightarrow A \preceq A$
Transitivity: $B \preceq C \Rightarrow A' \preceq C; \quad \forall A' \subseteq B$
$A \preceq B \Rightarrow A \sim A'$ for some $A' \subseteq B \Rightarrow A \preceq C$ □

Theorem 3.8. *Banach-Schroder Bernstein Theorem: Suppose we have an equivalence relation for equidecomposability $\sim$ under the action of some group G satisfying the following conditions:*
1. $A \sim B$ and $C \subseteq A$ implies there exists a bijection $f : A \to B : C \sim_G f(C)$
2. If $A_1 \cap A_2 = B_1 \cap B_2 = \emptyset$, and if $A_1 \sim B_1; A_2 \sim B_2$; then $A_1 \cup A_2 \sim B_1 \cup B_2$

$$\text{Then } A \preceq B \text{ and } B \preceq A \Rightarrow A \sim B$$

Proof. By Condition 1; we have bijections $f : A \to B_1$ and $g : B \to A_1$; $A_1 \subseteq A$; $B_1 \subseteq B$.

Let us define $C_0 = A \backslash A_1$ and $C_{n+1} = g(f(C_n))$' with $\bigcup_{n=1}^{\infty} C_n = C$

Then $x \in f(C) \Leftrightarrow g(x) \in C$.

We can see that $A \backslash C \subseteq A_1 = g(B)$. So $A \backslash C = g(B \backslash f(C))$

But, thanks to Condition 1, this shows that $A \backslash C$ is equidecomposable with $B \backslash f(C)$.
Then, by Condition 2: $A = A \backslash C \cup C \sim B \backslash f(C) \cup f(C) = B$

□

Theorem 3.9. *The action of G_3, the group of isometries on $\mathbb{R}^3$ satisfies the conditions of the Banach-Schroder Bernstein Theorem. Hence; $A \preceq_{G_3} B$ and $B \preceq_{G_3} A$ together imply that $A \sim_{G_3} B$; where A and B are some subsets of $\mathbb{R}^3$*

Proof.
Condition 1: If $A \sim B$, then we can dissect both sets into n pieces, such that each piece of A is congruent to exactly one piece of B. Every element of A is subjected to an isometry dependent on what 'piece' it belongs to, to map it to exactly one element of B. This is evidently a bijection: injectivity due to pairwise disjointness of the pieces, and the fact that isometry is distance preserving, so that two elements within a 'piece' cannot be mapped to one. Consider an arbitrary subset, C, of A. If we map C to B using the previous bijection, we can see that the mapping still depends on the 'n' isometries used to transport the pieces of A to B. Therefore, we can divide C into 'n' pieces, dependent on which 'piece' of A it belongs to, and transport it using the relevant isometry to B. C is therefore equidecomposable with its image under the bijection.
Condition 2: If $A_1 \sim B_1$ with m pieces, and $A_2 \sim B_2$ with n pieces, and if the requirements of condition 2 are satisfied, obviously $A_1 \cup A_2 \sim B_1 \cup B_2$ under the same partitioning and reassembly, using $m + n$ pieces.

□

Corollary 3.10. *Let G be a group acting on some set X such that the conditions of the Banach-Schroder-Bernstein Theorem are satisfied. $E \subseteq X$ is G-paradoxical if and only if there exists $A, B \subseteq X$ such that $A \cup B = E$ and $A \sim_G B \sim_G E$*

Proof. We know from Corollary 3.3 that E is G-paradoxical implies that there exist disjoint $A', B \subseteq E$ such that $A' \sim_G E \sim_G B$.
$A' \cup B = \emptyset \Rightarrow A \subseteq E \backslash B$
$A' \sim_G A' \Rightarrow A' \preceq_G E \backslash B \preceq_G E$
But $A' \sim_G E \Rightarrow E \backslash B \preceq_G A'$
The Banach-Schroder-Bernstein Theorem then ensures that $A' \sim_G E \backslash B$.
But then, taking $A = E \backslash B$, we have $A \cup B = E$ and $A \sim_G B \sim_G E$ □

We end the section on equidecomposability with a concrete example, which illustrates how it can be applied to bend geometrical intuition.

Theorem 3.11. *Let x be a point on S^1; the unit circle. $S^1 \backslash \{x\}$, the broken circle, is equidecomposable with S^1 under the SO_2; the group of rotations on $\mathbb{R}^2$*

Proof. First of all we may assume, without loss of generality, that the"break" in the circle lies on the point$(1,0)$, by suitably rotating the circle. Identify every point (x,y) in $\mathbb{R}^2$ with $(x+iy)$; the corresponding point in the complex plane. Then $S^1\backslash\{0\}$ corresponds to the set $\{z\in\mathbb{C}:|z|=1\}\backslash\{e^0\}$. Consider M, the countable set of points $\{e^{in}\}_{n\in\mathbb{N}^+}$. These all lie on the set on the complex plane corresponding to $S^1\backslash\{0\}$, as their respective moduli are all 1. The are pairwise disjoint as $e^{in}=e^{im}\Rightarrow e^{i(n-m)}=1\Rightarrow 2\pi\mid(n-m)$, which is absurd.
Let ρ be the anticlockwise rotation of S^1 by 1 radian. So $\rho(z)=e^{-iz}$.
Then $\rho(M)=M\cup\{e^0\}$.
Therefore, $S^1\backslash\{0\}=S^1\backslash(\{0\}\cup M)\cup M\sim_{SO_2}S^1\backslash(\{0\}\cup M)\cup\rho(M)=S^1$ □

4. The Banach Tarski Paradox

In our construction of the Hausdorff Paradox, we constructed a countable set $D\subset S$ such that $S^2\backslash D$ is paradoxical with respect to F_2, a constructed subgroup of rotations isomorphic to the free group on two generators. Our next step is to show that $S^2\backslash D\sim_G S^2$, for some group G enclosing F_2. Corollary 3.5 then ensures that S^2 is G paradoxical.

Lemma 4.1. *For any countable set $D\subset S^2$, there exists a rotation ρ of S^2 such that $D\cap\rho(D)=\emptyset$, and furthermore, for any $n,m\in\mathbb{N};\rho^n(D)\cap\rho^m(D)=\emptyset$*

Proof. Any axis of rotation fixes exactly two points on the sphere. Let us choose an axis such that no elements of D are fixed. Let us label the set of rotations on this axis: R.

For any $z\in\mathbb{N}$, $d\in D$; let:

$$H_{d,z}\subseteq R=\{h:h^z(d)\in D\}$$

Since D is countable, so is $H_{d,z}$, as there is a one to one correspondence between elements of D and $H_{d,z}$, namely, the map of rotations sending d to each respective element of D.

So $H=\bigcup_{z\in\mathbb{N}}\bigcup_{d\in D}H_{d,z}$ is countable, as the countable union of a countable number of countable sets

Since R is uncountable, being in bijection with the interval $[0,2\pi)$, we can see that we can choose a rotation ρ in $R\backslash H$
Therefore; $D\cap\rho(D)=\emptyset$, as the converse would imply that $\rho\in\bigcup_{d\in D}H_{d,1}$

Suppose, for some distinct $n, m \in \mathbb{N};$ we have $\rho^n(D) \cap \rho^m(D) \neq \emptyset$
Then, for some s_1, s_2 in $D : \rho^n(s_1) = \rho^m(s_2)$
Without loss of generality, suppose $n > m$
$\Rightarrow \rho^{n-m}(s_1)) = s_2 \in D$
$\Rightarrow \rho \in H_{s_1,(n-m)}$ $\perp$
$\Rightarrow \{\rho^k(D)\}_{k\in\mathbb{N}}$ is disjoint

□

Theorem 4.2. *For any countable set D, $S^2 \backslash D \sim_{SO_3} S^2$, with two pieces.*

Proof.
Taking ρ as in the previous lemma, we let $A = \bigcup_{k\in\mathbb{N}} \rho^k(D)$
Then $S^2\backslash A \cup A$ is a disjoint partition of S^2
Letting the identity rotation and ρ act on these two pieces respectively we get:

$$S^2\backslash A \cup A \sim_{SO_3} S^2\backslash A \cup \rho(A)$$

But $S^2\backslash A \cup \rho(A) = S^2\backslash D$
$\Rightarrow S^2 \sim_{SO_3} S^2\backslash D$

□

Corollary 4.3. *S^2 is paradoxical with respect to SO_3, the group of rotations.*

Proof. We have already shown that, for some countable set D, $S^2\backslash D$ is paradoxical. Corollary 3.5 ensures that any set equidecomposable with $S^2\backslash D$ is also paradoxical. □

Corollary 4.4. *The unit ball with centre point removed: $B^3\backslash\{0\}$, is paradoxical with respect to SO_3.*

Proof. This is in fact equivalent to the previous corollary. Since S^2 is paradoxical with respect to SO_3, we have two disjoint subsets A and B in S^2, such that each can be decomposed into a finite collection of pairwise disjoint subsets, and reassembled under isometries to form two copies of S^2.
For each $C \subseteq S^2$, let C^* be the set of radial lines in $B^3\backslash\{0\}$ containing C. This forms an equivalence relation between sets in S^2 and $B^3\backslash\{0\}$. Therefore, the sets A^* and B^*

in $B^3 \backslash \{0\}$ corresponding to A and B, yield paradoxical decompositions under the same isometries, over the set of radial lines with endpoints at S^2; ie $B^3 \backslash \{0\}$.

□

Lemma 4.5. *$B^3 \backslash \{0\}$ is equidecomposable under rotation with B^3*

Proof. Theorem 3.11 shows us that the circle is equidecomposable under rotation with the broken circle. Choose a broken circle $C' \subset B^3 \backslash \{0\}$, with its break at $\{0\}$, and let C be the full circle.

$$B^3 \backslash \{0\} = B^3 \backslash (\{0\} \cup C') \cup C' \sim_{SO_3} B^3 \backslash (\{0\} \cup C') \cup C = B^3$$

□

Theorem 4.6. *The Banach Tarski Paradox (weak form): B^3 is SO_3 paradoxical*

Proof. Since $B^3 \backslash \{0\}$ is paradoxical, and is equidecomposable with B^3, B^3 itself is paradoxical by Corollary 3.5. □

Note that in the standard formulation of the paradox, B^3 is shown to be paradoxical with respect to isometries. Since rotations are a subgroup of isometries, our result implies this. However, since SO_3 does not include translations, a visual interpretation of our version of the paradox is not as striking, as we obtain a unit ball that has been 'filled in twice over'. Trivially translating one covering of the unit ball yields two separate balls.

Note also in our proof that no use was made of metric of any form. Therefore, though our proof dealt notionally with the 'unit' ball, this theorem is equally valid for balls of any radius. As we have shown that one ball can be reformed into two balls, so we can show each of these balls can be respectively reformed into another two balls. By inductive application of this process, we can see that any finite number of balls can be so constructed from a single ball, thanks to the transitivity of the equidecomposability relation.

Corollary 4.7. *$\mathbb{R}^3$ is SO_3 paradoxical*

Proof. We constructed a paradoxical decomposition of $B^3 \backslash \{0\}$ by drawing radial lines from every point in S^2 to the origin, and identifying the lines with the points (Corollary 4.4). The paradoxicity of S^2 was then shown to be equivalent to that of $B^3 \backslash \{0\}$. If, rather than terminating the lines at the surface of the unit ball, we extended them infinitely, then we obtain the set $R^3 \backslash \{0\}$, and see that this set is also SO_3 paradoxical. We can then reproduce Lemma 4.5, substituting $B^3 \backslash \{0\}$ with $R^3 \backslash \{0\}$, to show that $R^3 \backslash \{0\}$ is equidecomposable with R^3. The result follows. □

We now want to extend the paradox just proved. We have overcome the intuition that isometry is a volume preserving property, by resorting to decomposing the unit ball into non-measurable subsets. It then logically follows that a much larger class of subsets of $\mathbb{R}^3$ could possibly be decomposed and reformed under isometry to yield subsets with a different volume. We now generalise our theorem accordingly.

Theorem 4.8. *Banach Tarski Paradox (Strong Form): All bounded subsets of R^3 with nonempty interior are equidecomposable under isometry.*

From Theorem 3.9; we know that G_3, the group of isometries on $\mathbb{R}^3$, satisfies the conditions of the Banach-Schroder-Bernstein Theorem. Using this and the fact that equidecomposability is an equivalence relation, it suffices to show that, for any bounded $C, D \subset \mathbb{R}^3$ with non-empty interior, $C \preceq_{G_3} D$ and $D \preceq_{G_3} C$.

Proof.
$\mathring{C} \neq \emptyset$. Pick $c \in \mathring{C}$

$$\Rightarrow \exists\, \epsilon > 0 : B(c, \epsilon) \subseteq C$$
$$\Rightarrow B(c, \epsilon) \preceq_G C$$

D is bounded,

$$\Rightarrow \exists r > 0, d \in D;\ \text{s.t. } D \subseteq B(d, r); D \preceq_G B(d, r)$$

If $B(d, r) \preceq_G B(a, \epsilon)$, then, from transitivity of the partial ordering relation, we can see that $D \preceq_G C$.

From the weak form of the Banach Tarski Paradox, we know that $B(c, \epsilon)$ is equidecomposable with two copies of itself. Each of these copies is then equidecomposable with two further copies of themselves. Repeating this inductively k times, we get that $B(c, \epsilon)$ is equidecomposable with 2^k copies of itself, due to the associativity of the equivalence relation for equidecomposability. Therefore, $B(c, \epsilon)$ is equidecomposable with a lattice of overlapping balls covering $B(d, R)$, we simply choose k sufficient to completely cover $B(d, R)$. Let us call this set of overlapping balls J.

$$B(b, R) \subseteq J \Rightarrow B(b, r) \preceq_G J$$

Then $C \succeq_G B(a, \epsilon) \succeq_G J \succeq_G B(b, r) \succeq_G D$. So $C \succeq_G D$.

In the above proof we used no property of C that was not shared by D, and vice versa. Therefore we can 'swap' the two sets, and repeat the proof to get that $D \succeq_G C$.

So C and D are equidecomposable. □

Now we have finally proved Banach Tarski in three dimensions, it would be natural to generalise the case into higher dimensions, which is indeed possible. However, in the context of this essay, it is a meaningless extension as there is no gap in intuition between the statement in three dimensions and higher dimensions

5. The Role of the Axiom of Choice in Banach Tarski

We used the axiom of choice in our proof of Banach Tarski. This does not necessarily imply that it is necessary to an arbitrary proof of the phenomenon. In this section we suggest that its necessity is likely.

Theorem 5.1. *Banach Tarski relies on Lebesgue non-measurable sets*

Proof. The Lebesgue measure is rotation and translation invariant by definition. Since pieces in the Banach Tarski decomposition of a unit ball are pairwise disjoint, the measure of their union (if it exists) must equal the sum of their measures, as must the measure of their union after any rotations or translations have been applied. This implies the measure of the unit ball in $\mathbb{R}^3$ is equivalent to twice the measure of itself, which is only possible if its measure is equal to zero. As it has non-empty interior, this is obviously not true. □

Proposition 5.2. *The existence of Lebesgue non-measurable sets depends on the axiom of choice*

This result is beyond the scope of this essay, and is not absolute. However, we can briefly elucidate the steps required. To do so requires a basic understanding of model theory, see [HOD97]. In summary, though, a structure S is a triple (U, σ, I) consisting of a set U, (the universe of the structure), equipped with well defined functions, and relations. For instance, elementary arithmetic is a model of $\mathbb{R}$ together with well defined functions such as addition, and the inequality relation. One can make a statement about the elements of S. The structure is said to model the statement if the statement is true given the functions and relations on S in the structure. So the statement: 'Every number has an inverse such that the sum of the number and its inverse under addition is zero' is true in the model of arithmetic. Solovay, in [SOL70], constructed a model in which all the statements of ZF are true, and for which there are no Lebesgue non-measurable subsets of $\mathbb{R}$. This would imply the existence of non-measurable subsets is a sole consequence of the axiom of choice, in ZFC. It is important to note, however, that this construction assumes consistency of ZFC, which, as I have discussed previously, is not proven.

.

Corollary 5.3. *The existence of the Banach Tarski Paradox is dependent upon the axiom of choice*

As I have previously stated, Banach Tarski is not truly paradoxical in the sense that only our intuition, rather than any form of consistency within ZFC, is compromised. Musing over whether the axiom of choice is 'correct' or not, is, therefore meaningless. Mathematics will contain results that challenge the intuition with or without it. We may merely distinguish that which requires it, and that which does not. In fact, its exclusion directly results in theorems such as the existence of sets whose cardinality cannot be compared. For a more complete treatise on the axiom and such results, the reader could refer to [HER06].

6. Geometrical Paradoxes without the Axiom of Choice

As a consequence of the previous section, the reader is tempted to believe that the axiom of choice is responsible for counterintuitive geometrical constructions, and therefore must be discarded in those mathematical results that are designed to have relevance in reality. I have already outlined in my introduction my reasons for disagreeing with this. To highlight the existence of paradoxical constructions independent of the axiom of choice, I present the Sierpinski-Mazurkiewicz paradox in this chapter, with its construction based on that found in [SU90].

Proposition 6.1. *Sierpinski-Mazurkiewicz Paradox: There exists a subset of* $\mathbb{R}^2$ *that is paradoxical with respect to* G_2*, the group of isometries on* $\mathbb{R}^2$

Lemma 6.2. *For any* $x, y \in G_2$*, let* $W_{xy}(x) \in G_2$ *be the set of reduced form words generated by* $\{x, y\}$ *and beginning with* x*. Then there exist isometries* $\tau, \rho \in G_2$ *such that:*

$$w_1 \in W_{\tau\rho}(\tau), w_2 \in W_{\tau\rho}(\rho) \Rightarrow w_1(0) \neq w_2(0)$$

Proof. The set of rotations on the complex plane is precisely the set of elements of the unit circle, and multiplication of a complex number by an element e^{ik} of the unit circle is equivalent to rotating it by k radians. There are uncountable elements on the unit circle, only countably which of can be algebraic, so there exists a transcendental point on the unit

circle (an element that is not a root of a non-trivial polynomial with rational coefficients). Let us pick such a point, and label it u, with the corresponding rotation $\rho(z) = uz$. Let $\tau(z) = z + 1$.
Suppose w, a word in G_2, ends in ρ. Then we can erase this final term without altering $w(0)$, as $\rho(0) = 0$. Pick arbitrary $w_1 \in W_{\tau\rho}(\tau), w_2 \in W_{\tau\rho}(\rho)$. Obtain t_1, t_2 by repeatedly deleting the end letters of w_1, w_2 if they are ρ, until τ is the last element of both words. Note that t_2 may be the empty word corresponding to the identity rotation. Then:

$$t_1 = \tau^{j_1}\rho^{j_2}\tau^{j_3}...\tau^{j_m} \qquad t_2 = \rho^{k_1}\tau^{k_2}\rho^{k_3}...\tau^{k_n} \qquad \{j_i\}, \{k_i\}, m, n \in \mathbb{N}$$

$$w_1(0) = t_1(0) = j_1 + j_3u^{j_2} + j_5u^{j_2+j_4} + ... + j_mu^{j_2+j_4+...} \tag{1}$$

$$w_2(0) = \begin{cases} t_2(0) = k_2u^{k_1} + k_4u^{k_1+k_3} + ... + k_nu^{k_1+k_3+...} : & t_2 \neq \emptyset \\ 0 : & t = \emptyset. \end{cases} \tag{2}$$

Now suppose $w_1(0) = w_2(0)$. Then $w_1(0) - w_2(0) = 0$. But, by the above two equations, $w_1(0) - w_2(0) = p(u)$, for some polynomial p. This contradicts transcendence of u, so our assumption is incorrect. □

Corollary 6.3. *L; the subset of G_2 generated by $\{\tau, \rho\}$ is isomorphic to the free semigroup on two generators*

Proof.
It suffices to show that any two distinct words $l_1, l_2 \in L$ do not represent the same rotation (see Definition 1.8).
Suppose $l_1 = l_2$; and the sequence of letters in one of these words is a segment of the other word. Then cancellation gives us a non-trivial word w' representing the identity rotation, and $w'\rho(0) = \rho(0)$; $w'(\tau(0) = \tau(0)$. One of these equations must contradict Lemma 6.2.

Suppose $l_1 = l_2$; and neither sequence of letters is a segment of the other. Then, if the leftmost letters in the alphabetical representation of this equation are the same, we cancel them, and repeat this operation until they are nonidentical. This gives us two words that represent the same rotation, so must send $\{0\}$ to the same point, yet begin with different letters, again contradicting Lemma 6.2. □

An incidental consequence of this result and Theorem 1.10 is that G_2 has a nonempty paradoxical subset. G_2 itself, however, is not paradoxical. In Chapter 2 we used Corollary 1.12, and the fact that G_2 contains the free group on two generators to prove paradoxicity of G_3. This is not valid for G_2 as L is not a subgroup. For most of the previous century, it was conjectured that the only paradoxical groups were those which contained the free group on two generators (See Proposition 1)

The previous two results leave us ready to prove Proposition 6.1:

Proof. Consider the L-orbit of 0; that is, the set of images of 0 under the action of L. Noting that $L = W_{\tau\rho}(\tau) \cup W_{\tau\rho}(\rho)$ and using the consequent fact that such an image must satisfy one of equations (1) and (2) derived from Lemma 6.2; we can explicitly define this set to be:

$$E = \{a_1 + a_2u^2 + a_3u^3 ... a_nu^n \quad n \in \mathbb{N}, a_i \in \mathbb{N}\}$$

We can see $\tau(E); \rho(E) \subset E$; and Lemma 6.2 shows us that $\tau(E) \cap \rho(E) = \emptyset$. Respectively applying the rotations τ^{-1} and ρ^{-1} to $\tau(E)$ and $\rho(E)$ shows us that E is paradoxical, with two pieces. □

Note that the Lebesgue Measure of E is 0, so the result does not contradict isometry invariance of the Lebesgue Measure.

7. Related Problems

No branch of Mathematics evolves linearly, and sure enough the solution of the Banach Tarski Paradox has spawned a network of supporting literature and related problems; some still open. This area of Mathematics is still flourishing, and I highlight that in this section, by expositing some of the most relevant recent results.

7.1. Tarski's Circle Squaring Problem.

Problem 7.1. *Is the unit circle (finitely) equidecomposable with the square with respect to the isometries of* $\mathbb{R}^2$*?*

This puzzle, posed by Tarski in 1925, foiled mathematicians for over 60 years. Laczkovitch [LAC90] finally solved the problem affirmatively in 1990. A proof that circle squaring is impossible by dissection of the circle into closed curves with nonempty interior was provided in 1963 [DUB63], and in fact the proof, like that of the Banach Tarski Paradox, relies on non-measurable subsets. It also utilises the axiom of choice. While the minimum number of pieces in the decomposition and doubling of the unit ball is 5; Laczovitch's decomposition of the circle used in the region of 10^{50} pieces. Another interesting fact is that the repartitioning is possible using only translations.

7.2. **Marczewski's Problem.**

Problem 7.2. *Can there exist a paradoxical decomposition of the unit ball using only pieces possessing the property of Baire?* [4]

This problem also has an affirmative resolution, as given in [RAN94]. More generally, any two bounded sets possessing the property of Baire are finitely equidecomposable with each other, using only pieces that also possess the property of Baire. This extension of the problem follows naturally in much the same way that we previously derived the strong form of Banach Tarski from the weak form.

7.3. **De Groot's problem.**

Problem 7.3. *Can the pieces in a paradoxical decomposition of one unit ball into two be chosen in such a way that they can be moved continuously (in $\mathbb{R}^3$) from their original to their final position without ever overlapping on one another?*

Posed in 1958, such a construction was given in 2005 [WIL05]. I find this as almost as counterintuitive as the paradox itself. Any such construction cannot use pieces which all possess the Baire property.

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[4] Recall a Baire set B is one which contains an open set U such that $B\backslash U$ is meagre, hence they are often termed 'almost open' sets. see [PRE08] for a more rigorous definition

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