Algebraic criteria for circuit realisations

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Abstract. This paper provides algebraic criteria for the number of inductors and capacitors which must be present in a realisation of a given electrical impedance function. The criteria are expressed in terms of the rank and signature of the associated Hankel, or Sylvester, or Bezoutian matrix, or equivalently in terms of an extended Cauchy index.

1 Introduction

The purpose of this paper is to provide algebraic criteria for the number of reactive elements that are needed in the realisation of a given impedance function in electrical circuits. The basis for these results is the paper of Youla and Tissi [20] which introduced the method of reactance extraction in network synthesis. There it was shown that the number of capacitors and inductors needed to realise a given driving-point behaviour is the same for *any* minimally reactive reciprocal realisation and is related to the number of positive and negative entries in a certain "reactance signature matrix" associated with the scattering matrix. In this paper we rework this result starting with the more familiar impedance function. We first relate the number of capacitors and inductors to the number of positive and negative eigenvalues of the Hankel matrix. In turn this is related to conditions on the Sylvester and Bezoutian matrices. The criteria for the latter matrices, and also in terms of an extended Cauchy index, are shown to be valid for non-proper impedances. The case of non-minimally reactive networks is also considered and the generalisation to multi-ports is discussed.

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Mit herzlichen Glückwünschen an Professor Uwe Helmke
anlässlich seines 60. Geburtstags.

2 Notation

We denote the *rank* of a matrix by $r(\cdot)$ and the *determinant* of a square matrix by $|\cdot|$. For a real symmetric matrix we denote the number of strictly positive and strictly

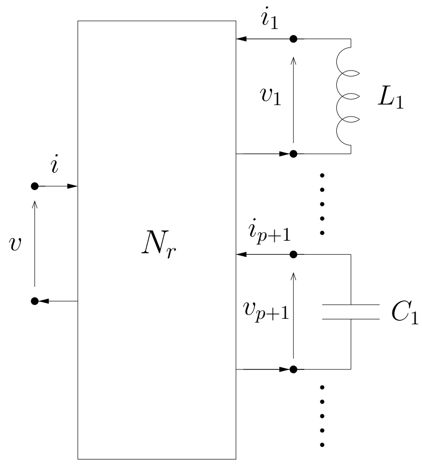

Figure 1: One-port network N with reactive elements extracted.

negative eigenvalues by $\pi(\cdot)$ and $\nu(\cdot)$ respectively. The *signature* $\sigma(\cdot)$ of a real symmetric matrix is defined by $\sigma(\cdot) = \pi(\cdot) - \nu(\cdot)$. Let $x_1, \ldots, x_r$ be a sequence of non-zero real numbers. We define $\mathbf{P}(x_1, \ldots, x_r)$ to be the number of permanences of sign and $\mathbf{V}(x_1, \ldots, x_r)$ to be the number of variations of sign in the sequence $x_1, \ldots, x_r$. We denote the set of real-rational functions in the variable s by $\mathbb{R}(s)$. The subset of *proper* rational functions, denoted by $\mathbb{R}_p(s)$, are those which are bounded at $s = \infty$. We similarly denote the set of real-rational matrix functions with r rows and c columns by $\mathbb{R}^{r\times c}(s)$ and the corresponding subset of proper real-rational matrix functions by $\mathbb{R}_p^{r\times c}(s)$. We denote the *McMillan degree* [3, Section 3.6] of a function $F(s) \in \mathbb{R}^{r\times c}(s)$ by $\delta(F(s))$. If $F(s) = a(s)/b(s) \in \mathbb{R}(s)$ with $a(s)$ and $b(s)$ coprime then $\delta(F(s)) = \max\{\deg(a(s)), \deg(b(s))\}$. The *extended Cauchy index* of a rational function or a symmetric rational matrix function (see Definitions 5 and 12) is denoted by $\gamma(F(s))$. We call a factorisation of a function $F(s) \in \mathbb{R}^{r\times c}(s)$ into the form $F(s) = B^{-1}(s)A(s)$ for $A(s)$, $B(s)$ real polynomial matrices in s a *left matrix factorisation*. For a symmetric matrix $F(s) \in \mathbb{R}^{m\times m}(s)$ with left matrix factorisation $F(s) = B^{-1}(s)A(s)$ we denote the Bezoutian by $\mathcal{B}(B, A)$ (see Sections 6 and 9). We denote by $X \dotplus Y$ the block diagonal matrix with diagonal blocks X and Y.

3 Reactance extraction and the Hankel matrix

We begin with a function $Z(s) \in \mathbb{R}_p(s)$ with $\delta(Z(s)) = n$. Suppose $Z(s)$ is the impedance of a one-port network N containing only transformers, resistors and reactive elements (inductors and capacitors) with positive values, hereafter referred to as a *reciprocal network*. Then N contains no fewer than n reactive elements [3, Theorem 4.4.3], and is called *minimally reactive* if it contains exactly this many.

Suppose that N contains exactly p inductors and q capacitors and is minimally reactive, so $p + q = n$. Using the procedure of reactance extraction [20] N takes the form of Figure 1 where the network N_r possesses a hybrid matrix M such that

$$\begin{bmatrix} v \\ \mathbf{v_a} \\ \mathbf{i_b} \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \begin{bmatrix} i \\ \mathbf{i_a} \\ \mathbf{v_b} \end{bmatrix}, \tag{1}$$

where $\mathbf{i_a} = \left[i_1, \ldots, i_p\right]^\top$ is the vector of (Laplace-transformed) currents through the inductors in N with $\mathbf{v_a}$ the corresponding voltages, $\mathbf{v_b} = \left[v_{p+1}, \ldots, v_{p+q}\right]^\top$ is the vector of (Laplace-transformed) voltages across the capacitors in N with $\mathbf{i_b}$ the corresponding currents, and the matrix M is partitioned compatibly with the pertinent vectors. The existence of a hybrid matrix in the form (1) follows from [3, Section 4.4] and is discussed in greater detail in Section 8 of this paper. Since N_r is a reciprocal network then, by [3, Theorem 2.8.1],

$$(1 \dotplus \Sigma) M = M^\top (1 \dotplus \Sigma), \tag{2}$$

where $\Sigma = \left(I_p \dotplus -I_q\right)$. When terminated on the reactive elements we have

$$\begin{bmatrix} \mathbf{v_a} \\ \mathbf{i_b} \end{bmatrix} = -s\Lambda \begin{bmatrix} \mathbf{i_a} \\ \mathbf{v_b} \end{bmatrix},$$

where $\Lambda = \operatorname{diag}\{L_1, \ldots, L_p, C_1, \ldots, C_q\}$. Then it can readily be seen that $Z(s) = J + H(sI - F)^{-1} G$ where

$$F = -\Lambda^{-1} \begin{bmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{bmatrix} \in \mathbb{R}^{n \times n}, \tag{3}$$

$$G = -\Lambda^{-1} \begin{bmatrix} M_{21} \\ M_{31} \end{bmatrix} \in \mathbb{R}^{n \times 1}, \tag{4}$$

$$H = \begin{bmatrix} M_{12} & M_{13} \end{bmatrix} \in \mathbb{R}^{1 \times n}, \tag{5}$$

$$J = M_{11} \in \mathbb{R}, \tag{6}$$

and, since $\Sigma^2 = I_n$, and Σ and Λ are both diagonal, from (2) we have

$$F = \Lambda^{-1} \Sigma F^\top \Sigma \Lambda, \tag{7}$$

$$G = -\Lambda^{-1} \Sigma H^\top. \tag{8}$$

Consider the controllability and observability matrices

$$V_c = \left[G, FG, \ldots, F^{n-1} G\right], \tag{9}$$

$$V_o = [H^\top, F^\top H^\top, \ldots, \left(F^\top\right)^{n-1} H^\top]^\top. \tag{10}$$

Since $\delta(Z(s)) = n$ the state-space realisation (3-6) must be controllable and observable and hence V_o and V_c both have rank n. Furthermore from (7,8) we have

$$V_c = -\Lambda^{-1} \Sigma V_o^\top. \tag{11}$$

We introduce the Hankel matrix

$$\mathcal{H}_n = V_o V_c = \begin{bmatrix} h_0 & h_1 & \ldots & h_{n-1} \\ h_1 & h_2 & \ldots & h_n \\ \vdots & \vdots & \ddots & \vdots \\ h_{n-1} & h_n & \ldots & h_{2n-2} \end{bmatrix}, \tag{12}$$

where $h_i = HF^iG$ for $i = 0,1,2,\ldots$ are the Markov parameters, which are also directly defined from the Laurent expansion

$$Z(s) = h_{-1} + \frac{h_0}{s} + \frac{h_1}{s^2} + \frac{h_2}{s^3} + \ldots. \tag{13}$$

It follows from (11) that

$$\mathcal{H}_n = V_o\left(-\Lambda^{-1}\Sigma\right)V_o^\top. \tag{14}$$

From (14) and Sylvester's law of inertia [15] we deduce the following.

Theorem 1. *Let $Z(s) \in \mathbb{R}_p(s)$ with $\delta(Z(s)) = n$ and let $\mathcal{H}_n$ be as in (12) for $Z(s)$ as in (13). If $Z(s)$ is the impedance of a reciprocal network containing exactly p inductors and q capacitors with $p+q = n$ then $\pi(\mathcal{H}_n) = q$ and $\nu(\mathcal{H}_n) = p$.*

Define the infinite Hankel matrix

$$\mathcal{H} = \begin{bmatrix} h_0 & h_1 & h_2 & \ldots \\ h_1 & h_2 & h_3 & \ldots \\ h_2 & h_3 & h_4 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{15}$$

and the corresponding finite Hankel matrices

$$\mathcal{H}_k = \begin{bmatrix} h_0 & h_1 & \ldots & h_{k-1} \\ h_1 & h_2 & \ldots & h_k \\ \vdots & \vdots & \ddots & \vdots \\ h_{k-1} & h_k & \ldots & h_{2k-2} \end{bmatrix}, \tag{16}$$

for $k = 1,2,\ldots$. Then it is known that $\mathcal{H}$ has finite rank equal to n and $|\mathcal{H}_n| \neq 0$ [10, p. 206-7]. From (14) and [9, Theorem 24, p. 343] we have the following.

Theorem 2. *Let $Z(s) \in \mathbb{R}_p(s)$ with $\delta(Z(s)) = n$ and let $\mathcal{H}_k$ be as in (16) for $Z(s)$ as in (13). If $Z(s)$ is the impedance of a reciprocal network containing exactly p inductors and q capacitors with $p+q = n$ then $|\mathcal{H}_n| \neq 0$, $|\mathcal{H}_k| = 0$ for $k > n$, and*

$$q = \mathbf{P}(1, |\mathcal{H}_1|, \ldots, |\mathcal{H}_n|), \tag{17}$$
$$p = \mathbf{V}(1, |\mathcal{H}_1|, \ldots, |\mathcal{H}_n|). \tag{18}$$

In any subsequence of zero values, $|\mathcal{H}_k| \neq 0$, $|\mathcal{H}_{k+1}| = |\mathcal{H}_{k+2}| = \ldots = 0$, signs are assigned to the zero values as follows: $\operatorname{sign}\left(|\mathcal{H}_{k+j}|\right) = (-1)^{\frac{j(j-1)}{2}} \operatorname{sign}(|\mathcal{H}_k|)$.

4 The Cauchy index and the Sylvester matrix

The *Cauchy index* of a real-rational function $F(s)$ between limits $-\infty$ and $+\infty$, denoted $I_{-\infty}^{+\infty}F(s)$, is the difference between the number of jumps of $F(s)$ from $-\infty$ to $+\infty$ and the number of jumps from $+\infty$ to $-\infty$ as s is increased in $\mathbb{R}$ from $-\infty$ to $+\infty$. From [10, Theorem 9, p. 210], if $F(s) \in \mathbb{R}_p(s)$ then $I_{-\infty}^{+\infty}F(s)$ is equal to the signature of the corresponding Hankel matrix. From Theorem 1 we deduce the following.

Theorem 3. *Let $Z(s) \in \mathbb{R}_p(s)$ be the impedance of a reciprocal network containing exactly p inductors and q capacitors and with $p+q=\delta(Z(s))$. Then*

$$q-p = I_{-\infty}^{+\infty} Z(s).$$

We now write

$$Z(s) = \frac{a(s)}{b(s)} = \frac{a_n s^n + a_{n-1}s^{n-1} + \ldots + a_0}{b_n s^n + b_{n-1}s^{n-1} + \ldots + b_0}. \tag{19}$$

Multiplying by $b(s)$ in (13) and equating coefficients of s we obtain

$$\begin{aligned} h_{-1}b_n &= a_n, \\ h_{-1}b_{n-1} + h_0 b_n &= a_{n-1}, \\ &\vdots \\ h_{-1}b_0 + h_0 b_1 + \ldots + h_{n-2}b_{n-1} + h_{n-1}b_n &= a_0, \\ h_r b_0 + h_{r+1}b_1 + \ldots + h_{r+n-1}b_{n-1} + h_{r+n}b_n &= 0, \quad (r = 0,1,\ldots). \end{aligned}$$

Define the matrices

$$\mathcal{S}_{2k} = \begin{bmatrix} b_n & b_{n-1} & \ldots & b_{n-k+1} & b_{n-k} & \ldots & b_{n-2k+1} \\ a_n & a_{n-1} & \ldots & a_{n-k+1} & a_{n-k} & \ldots & a_{n-2k+1} \\ 0 & b_n & \ldots & b_{n-k+2} & b_{n-k+1} & \ldots & b_{n-2k+2} \\ 0 & a_n & \ldots & a_{n-k+2} & a_{n-k+1} & \ldots & a_{n-2k+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & b_n & b_{n-1} & \ldots & b_{n-k} \\ 0 & 0 & \ldots & a_n & a_{n-1} & \ldots & a_{n-k} \end{bmatrix}, \tag{20}$$

for $k = 1,2,\ldots$, in which we put $a_j = 0$, $b_j = 0$ for $j < 0$. Following [10, p. 214] we observe that $\mathcal{S}_{2k} = \Gamma_{2k}U_{2k}$ where

$$\Gamma_{2k} = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ h_{-1} & h_0 & \ldots & h_{k-2} & h_{k-1} & \ldots & h_{2k-2} \\ 0 & 1 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & h_{-1} & \ldots & h_{k-3} & h_{k-2} & \ldots & h_{2k-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & h_{-1} & h_0 & \ldots & h_{k-1} \end{bmatrix},$$

$$U_{2k} = \begin{bmatrix} b_n & b_{n-1} & b_{n-2} & \ldots & b_{n-2k+1} \\ 0 & b_n & b_{n-1} & \ldots & b_{n-2k+2} \\ 0 & 0 & b_n & \ldots & b_{n-2k+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & b_n \end{bmatrix}.$$

Since a sequence of $k(k-1)$ pairwise row permutations carries Γ_{2k} into a block lower triangular matrix with diagonal blocks I_k and $\mathcal{H}_k$ then

$$|\mathcal{S}_{2k}| = b_n^{2k}|\mathcal{H}_k|. \tag{21}$$

It may be observed that $|\mathcal{S}_{2n}|$ is the Sylvester resultant of $a(s)$ and $b(s)$, which is well known to be non-zero when $a(s)$ and $b(s)$ are coprime. Accordingly we will refer to the matrices $\mathcal{S}_{2k}$ in (20) as *Sylvester matrices*. If $Z(s) \in \mathbb{R}_p(s)$ then $b_n \neq 0$ and from (21) and Theorem 2 we obtain the following.

Theorem 4. *Let $Z(s) \in \mathbb{R}_p(s)$ with $\delta(Z(s)) = n$ and let $|\mathcal{S}_{2k}|$ be as in (20) for $Z(s)$ as in (19). If $Z(s)$ is the impedance of a reciprocal network containing exactly p inductors and q capacitors with $p+q=n$ then $|\mathcal{S}_{2n}| \neq 0$, $|\mathcal{S}_{2k}| = 0$ for $k > n$, and*

$$q = \mathbf{P}(1, |\mathcal{S}_2|, |\mathcal{S}_4|, \ldots, |\mathcal{S}_{2n}|),$$
$$p = \mathbf{V}(1, |\mathcal{S}_2|, |\mathcal{S}_4|, \ldots, |\mathcal{S}_{2n}|).$$

In any subsequence of zero values, $|\mathcal{S}_{2k}| \neq 0$, $|\mathcal{S}_{2(k+1)}| = |\mathcal{S}_{2(k+2)}| = \ldots = 0$, signs are assigned to the zero values as follows: $\operatorname{sign}\left(|\mathcal{S}_{2(k+j)}|\right) = (-1)^{\frac{j(j-1)}{2}} \operatorname{sign}(|\mathcal{S}_{2k}|)$.

We remark that Theorem 4 still holds when the polynomials $a(s)$ and $b(s)$ in (19) are not coprime providing we replace n with $r = \delta(a(s)/b(s))$ in the above theorem statement. Indeed the conditions $|\mathcal{S}_{2r}| \neq 0$ and $|\mathcal{S}_{2k}| = 0$ for all $k > r$ hold if and only if the function $Z(s)$ in (19) has $\delta(Z(s)) = r$ or equivalently the polynomials $a(s)$ and $b(s)$ have exactly $n-r$ roots in common.

5 Non-proper impedances and the extended Cauchy index

We consider the extension of the previous results to general rational functions (without the assumption of properness). We first introduce the following.

Definition 5. For $F(s) \in \mathbb{R}(s)$ we define the *extended Cauchy index* $\gamma(F(s))$ to be the difference between the number of jumps of $F(s)$ from $-\infty$ to $+\infty$ and the number of jumps from $+\infty$ to $-\infty$ as s increases from a point a through $+\infty$ and then from $-\infty$ to a again, for any $a \in \mathbb{R}$ which is not a pole of $F(s)$.

If $F(s)$ is proper or has a pole of even multiplicity at $s = \infty$ then $\gamma(F(s)) = I_{-\infty}^{+\infty} F(s)$. If $F(s)$ is non-proper and has a pole of odd multiplicity at $s = \infty$ then $\gamma(F(s))$ differs from $I_{-\infty}^{+\infty} F(s)$ by ± 1. Note that Definition 5 does not depend on the choice of a. It is straightforward to verify the following.

Lemma 6. *Let $F(s)$, $F_1(s)$, $F_2(s) \in \mathbb{R}(s)$. Then*

1. *$\gamma(F(s)) = -\gamma(1/F(s))$.*
2. *If $F(s) = F_1(s) + F_2(s)$ and $\delta(F(s)) = \delta(F_1(s)) + \delta(F_2(s))$ then $\gamma(F(s)) = \gamma(F_1(s)) + \gamma(F_2(s))$.*

Now suppose that a non-proper $Z(s)$ with $\delta(Z(s)) = n$ is the impedance of a minimally reactive reciprocal network containing p inductors and q capacitors. Then $1/Z(s)$ is (strictly) proper and is the admittance of the network. Again following [3, Section 4.4, Theorem 2.8.1], reactance extraction provides a hybrid matrix M satisfying (1) with v and i interchanged, and with (2) satisfied for $\Sigma = \left(-I_p \dotplus I_q\right)$. If we now form the Hankel matrix $\mathcal{H}_n^\dagger$ corresponding to $1/Z(s)$ we can deduce that

$$p - q = \sigma(\mathcal{H}_n^\dagger) = \gamma(1/Z(s)),$$

where we have used the same reasoning as for Theorem 1 (noting the change in sign due to the change in sign in Σ) and the fact that the extended Cauchy index for a proper rational function is equal to the signature of the corresponding Hankel matrix [10, p. 210]. Hence using Lemma 6(1.) and combining with Theorem 3 for the case that $Z(s)$ is proper we obtain the following result.

Theorem 7. *Let $Z(s) \in \mathbb{R}(s)$ be the impedance of a reciprocal network containing exactly p inductors and q capacitors and with $p+q = \delta(Z(s))$. Then*

$$q - p = \gamma(Z(s)).$$

We further consider a non-proper $Z(s)$. As in Section 3 we can form Hankel matrices $|\mathcal{H}_k^{\dagger}|$ corresponding to $1/Z(s)$. It can then be seen that Theorem 2 holds with $Z(s)$ replaced by $1/Z(s)$, the expressions for q and p in (17,18) interchanged, and $|\mathcal{H}_k|$ replaced everywhere by $|\mathcal{H}_k^{\dagger}|$. Now if $Z(s)$ is written in the form (19) then $a_n \neq 0$ and we can define Sylvester matrices $\mathcal{S}_{2k}^{\dagger}$ corresponding to $1/Z(s)$. As in Section 4 it follows that

$$|\mathcal{S}_{2k}^{\dagger}| = a_n^{2k} |\mathcal{H}_k^{\dagger}|. \tag{22}$$

We further note that $\mathcal{S}_{2k}^{\dagger}$ differs from $\mathcal{S}_{2k}$ by the interchange of row i with row $i+1$ for i odd. Therefore

$$|\mathcal{S}_{2k}^{\dagger}| = (-1)^k |\mathcal{S}_{2k}|. \tag{23}$$

Combining the modified form of Theorem 2 with (22) and (23) we obtain the following.

Theorem 8. *Theorem 4 (and its subsequent remark) holds for any $Z(s) \in \mathbb{R}(s)$.*

6 The Bezoutian matrix

Let $Z(s) \in \mathbb{R}(s)$ be written as in (19). The *Bezoutian* matrix is a symmetric matrix $\mathcal{B} = \mathcal{B}(b,a)$ whose elements $\mathcal{B}_{ij}$ satisfy

$$a(w)b(z) - b(w)a(z) = \sum_{i=1}^{n}\sum_{j=1}^{n} \mathcal{B}_{ij} z^{i-1}(z-w)w^{j-1}. \tag{24}$$

If $Z(s) \in \mathbb{R}_p(s)$ then, for $\mathcal{H}_k$ as in (16) with $Z(s)$ written as in (13), the matrix $\mathcal{B}(b,a)$ is congruent to $\mathcal{H}_n$ [8, equation 8.58]. It follows that $\gamma(Z(s)) = \sigma(\mathcal{H}_n) = \sigma(\mathcal{B}(b,a))$ and $\delta(Z(s)) = r(\mathcal{H}_n) = r(\mathcal{B}(b,a))$, these relationships holding irrespective of whether $a(s)$ and $b(s)$ are coprime. If $Z(s)$ is not proper then, since $b(s)/a(s)$ is proper and $\mathcal{B}(b,a) = -\mathcal{B}(a,b)$, we have that $\gamma(Z(s)) = -\gamma(1/Z(s)) = -\sigma(\mathcal{B}(a,b)) = \sigma(\mathcal{B}(b,a))$ and $\delta(Z(s)) = r(\mathcal{B}(a,b)) = r(\mathcal{B}(b,a))$. There is also a close relationship between the Bezoutian matrix and the Sylvester matrix. Let $Z(s)$ be as in (19) and let $\mathcal{B}_k$ be the matrix formed from the final k rows and columns of $\mathcal{B}(b,a)$, i.e.

$$\mathcal{B}_k = (\mathcal{B}_{ij})_{i,j=n-k+1}^{n}, \tag{25}$$

for $k = 1,2,\dots,n$. Define matrices $T, P_{11}, P_{12}, P_{21}, P_{22} \in \mathbb{R}^{k\times k}$ where

$$T = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

and

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} a_{n-k} & \cdots & a_{n-2k+1} & b_{n-k} & \cdots & b_{n-2k+1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & \cdots & a_{n-k} & b_{n-1} & \cdots & b_{n-k} \\ a_n & \cdots & a_{n-k+1} & b_n & \cdots & b_{n-k+1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & a_n & 0 & \cdots & b_n \end{bmatrix},$$

in which we put $a_j = 0$, $b_j = 0$ for $j < 0$. Then, following [8, Theorem 8.44], the matrices P_{21} and P_{22} commute and, using a Gohberg-Semencul formula [12, Theorem 5.1], we find

$$|P| = |P_{11}P_{22} - P_{12}P_{21}| = |\mathcal{B}_k||T|.$$

Since a sequence of $k(k-1)/2$ pairwise column permutations carries T into I_k, and a sequence of $k(3k-1)/2$ pairwise column permutations followed by $k(2k-1)$ pairwise row permutations carries P into $\mathcal{S}_{2k}^{\mathsf{T}}$, it follows that

$$|\mathcal{S}_{2k}| = |\mathcal{B}_k|,$$

for $k = 1,2,\dots,n$. Theorems 7 and 8 then lead to the following result.

Theorem 9. *Let $Z(s) \in \mathbb{R}(s)$ be as in (19) with $\delta(Z(s)) = n$. Further let $\mathcal{B}_k$ be as in (25) for $\mathcal{B}_{ij}$, $\mathcal{B}(b,a)$ defined via (24). If $Z(s)$ is the impedance of a reciprocal network containing exactly p inductors and q capacitors with $p+q = n$ then*

$$q = \frac{1}{2}\left(\delta(Z(s)) + \gamma(Z(s))\right) = \pi(\mathcal{B}(b,a)) = \mathbf{P}(1, |\mathcal{B}_1|, \dots, |\mathcal{B}_n|),$$
$$p = \frac{1}{2}\left(\delta(Z(s)) - \gamma(Z(s))\right) = \nu(\mathcal{B}(b,a)) = \mathbf{V}(1, |\mathcal{B}_1|, \dots, |\mathcal{B}_n|).$$

In any subsequence of zero values, $|\mathcal{B}_k| \neq 0, |\mathcal{B}_{k+1}| = |\mathcal{B}_{k+2}| = \dots = 0$ signs are assigned to the zero values as follows: $\operatorname{sign}\left(|\mathcal{B}_{k+j}|\right) = (-1)^{\frac{j(j-1)}{2}} \operatorname{sign}(|\mathcal{B}_k|)$.

We remark that the above theorem still holds when the polynomials $a(s)$ and $b(s)$ are not coprime providing we replace n with $r = \delta(a(s)/b(s))$ in the theorem statement.

7 Biquadratic functions

Despite their apparent simplicity the realisation of biquadratic functions has been much studied by circuit theorists. Accordingly we write down explicitly the conditions obtained in this paper which apply to this class. Let

$$Z(s) = \frac{a_2 s^2 + a_1 s + a_0}{b_2 s^2 + b_1 s + b_0}. \tag{26}$$

The Sylvester matrix $\mathcal{S}_4$ takes the form

$$\mathcal{S}_4 = \begin{bmatrix} b_2 & b_1 & b_0 & 0 \\ a_2 & a_1 & a_0 & 0 \\ 0 & b_2 & b_1 & b_0 \\ 0 & a_2 & a_1 & a_0 \end{bmatrix},$$

and we have

$$|\mathcal{S}_2| = b_2a_1 - b_1a_2,$$
$$|\mathcal{S}_4| = (b_2a_1 - b_1a_2)(b_1a_0 - b_0a_1) - (b_2a_0 - b_0a_2)^2.$$

The realisability conditions implied by Theorem 8 are shown in Table 1. Note that $|\mathcal{S}_4| > 0$ implies $|\mathcal{S}_2| \neq 0$. The conditions take the identical form if Theorem 9 is used together with the Bezoutian

$$\mathcal{B}_2 = \begin{bmatrix} b_1a_0 - a_1b_0 & b_2a_0 - a_2b_0 \\ b_2a_0 - a_2b_0 & b_2a_1 - a_2b_1 \end{bmatrix}.$$

In Table 1 it may be observed that whether the reactive elements are of the same kind, or of different kind, is determined by the sign of the resultant $|\mathcal{S}_4|$. This fact is stated by Foster [7] but no proof is provided, as noted by Kalman [14]. Also, for the case that $|\mathcal{S}_4| > 0$, [7] differentiates the 2 cases in Table 1 according to $\text{sign}(b_2a_0 - a_2b_0)$ rather than $\text{sign}(|\mathcal{S}_2|)$, which is easily shown to be equivalent.

Table 1 does not contain any information about synthesis, namely whether a reciprocal realisation exists for a given impedance function $Z(s)$, only the properties that a minimally reactive reciprocal realisation must satisfy if it does exist. It is well known that a function is realisable by a passive network if and only if it is positive-real. For the biquadratic (26) this is equivalent to

$$b_1a_1 - \left(\sqrt{b_0a_2} - \sqrt{b_2a_0}\right)^2 \geq 0,$$

and all coefficients in (26) have the same sign. Under this condition it is known that minimally reactive reciprocal realisations always exist [20], [3] and that transformers are not needed if $|\mathcal{S}_4| > 0$ (see Section 10). On the other hand, transformers are needed for some functions if $|\mathcal{S}_4| < 0$ [16]. Results on the classification of transformerless, minimally reactive reciprocal realisations of the biquadratic can be found in [13].

	$\lvert\mathcal{S}_2\rvert > 0$	$\lvert\mathcal{S}_2\rvert < 0$	$\lvert\mathcal{S}_2\rvert = 0$
$\lvert\mathcal{S}_4\rvert > 0$	$(0,2)$	$(2,0)$	-
$\lvert\mathcal{S}_4\rvert < 0$	$(1,1)$	$(1,1)$	$(1,1)$
$\lvert\mathcal{S}_4\rvert = 0$	$(0,1)$	$(1,0)$	$(0,0)$

Table 1: The number of reactive elements (# inductors, # capacitors) in a minimally reactive reciprocal realisation of a biquadratic.

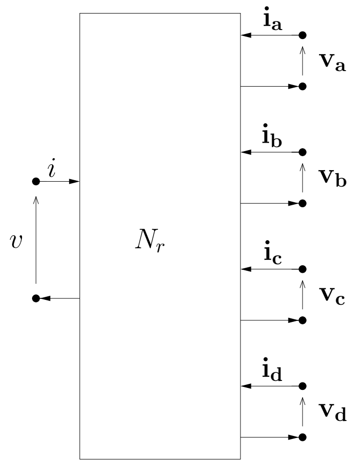

Figure 2: The network N_r obtained by removing all reactive elements from N.

8 Non-minimally reactive networks

Youla and Tissi use the scattering matrix formalism to establish lower bounds on the number of capacitors and inductors which are needed in reciprocal realisations (possibly non-minimally reactive) of a given scattering matrix [20, Theorem 2]. In this section we derive an equivalent result using the reactance extraction procedure as described in Anderson and Vongpanitlerd [3].

Let $Z(s) \in \mathbb{R}_p(s)$ be the impedance matrix of a one-port reciprocal network N containing exactly p inductors and q capacitors. Using the procedure in [3, Section 4.4], upon removal of the reactive elements in N we are left with the network N_r in Fig. 2 possessing a hybrid matrix M [3, equation 4.4.56] such that

$$\begin{bmatrix} v \\ \mathbf{v_a} \\ \mathbf{i_b} \\ \mathbf{i_c} \\ \mathbf{v_d} \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} & M_{15} \\ M_{21} & M_{22} & M_{23} & M_{24} & M_{25} \\ M_{31} & M_{32} & M_{33} & M_{34} & M_{35} \\ -M_{14}^{\top} & -M_{24}^{\top} & -M_{34}^{\top} & 0 & 0 \\ -M_{15}^{\top} & -M_{25}^{\top} & -M_{35}^{\top} & 0 & 0 \end{bmatrix} \begin{bmatrix} i \\ \mathbf{i_a} \\ \mathbf{v_b} \\ \mathbf{v_c} \\ \mathbf{i_d} \end{bmatrix},$$

where $(\mathbf{i_a}, \mathbf{v_a}), \ldots, (\mathbf{i_d}, \mathbf{v_d})$ are pairs of Laplace-transformed vectors of currents and voltages of dimensions p', q', $p-p'$, $q-q'$ respectively, and M is partitioned compatibly with the pertinent vectors. The network N is obtained upon terminating the ports corresponding to $(\mathbf{i_a}, \mathbf{v_a})$, $(\mathbf{i_c}, \mathbf{v_c})$ with inductors and the ports $(\mathbf{i_b}, \mathbf{v_b})$, $(\mathbf{i_d}, \mathbf{v_d})$ with capacitors. Then we have

$$\begin{bmatrix} \mathbf{v_a} \\ \mathbf{i_b} \end{bmatrix} = -s \begin{bmatrix} \mathcal{L}_2 & 0 \\ 0 & \mathcal{C}_3 \end{bmatrix} \begin{bmatrix} \mathbf{i_a} \\ \mathbf{v_b} \end{bmatrix},$$
$$\begin{bmatrix} \mathbf{v_c} \\ \mathbf{i_d} \end{bmatrix} = -s \begin{bmatrix} \mathcal{L}_4 & 0 \\ 0 & \mathcal{C}_5 \end{bmatrix} \begin{bmatrix} \mathbf{i_c} \\ \mathbf{v_d} \end{bmatrix},$$

where $\mathcal{L}_2 = \text{diag}\left(L_1, \ldots, L_{p'}\right)$, $\mathcal{C}_3 = \text{diag}\left(C_1, \ldots, C_{q'}\right)$, $\mathcal{L}_4 = \text{diag}\left(L_{p'+1}, \ldots, L_p\right)$ and $\mathcal{C}_5 = \text{diag}\left(C_{q'+1}, \ldots, C_q\right)$. It follows that equations (4.4.60) and (4.4.61) in [3, p. 195] must hold.

Since N_r is reciprocal then, by [3, Theorem 2.8.1],

$$\left(1 \dotplus I_{p'} \dotplus -I_{q'} \dotplus -I_{p-p'} \dotplus I_{q-q'}\right) M = M^\top \left(1 \dotplus I_{p'} \dotplus -I_{q'}, \dotplus -I_{p-p'} \dotplus I_{q-q'}\right). \tag{27}$$

which implies that all entries in M_{15}, M_{25} and M_{34} are zero. Furthermore since $Z(s)$ is proper we require $M_{14} = 0$. It may then be verified that $Z(s)$ has a state-space realisation with state vector $\left[\mathbf{i_a}^\top, \mathbf{v_b}^\top\right]^\top$ with dimension $n = p' + q'$ and with $Z(s) = J + H\left(sI - F\right)^{-1} G$ where

$$F = -R \begin{bmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{bmatrix} \in \mathbb{R}^{n \times n}, \tag{28}$$

$$G = -R \begin{bmatrix} M_{21} \\ M_{31} \end{bmatrix} \in \mathbb{R}^{n \times 1}, \tag{29}$$

$$H = \begin{bmatrix} M_{12} & M_{13} \end{bmatrix} \in \mathbb{R}^{1 \times n}, \tag{30}$$

$$J = M_{11} \in \mathbb{R}. \tag{31}$$

Here

$$R = \begin{bmatrix} R_{11} & 0 \\ 0 & R_{22} \end{bmatrix},$$

with

$$R_{11} = \left(\mathcal{L}_2 + M_{24}\mathcal{L}_4 M_{24}^\top\right)^{-1} \in \mathbb{R}^{p' \times p'},$$
$$R_{22} = \left(\mathcal{C}_3 + M_{35}\mathcal{C}_5 M_{35}^\top\right)^{-1} \in \mathbb{R}^{q' \times q'},$$

where existence of $R_{11} > 0$ and $R_{22} > 0$ is guaranteed since both $\left(\mathcal{L}_2 + M_{24}\mathcal{L}_4 M_{24}^\top\right)$ and $\left(\mathcal{C}_3 + M_{35}\mathcal{C}_5 M_{35}^\top\right)$ are positive definite.

Let $\Sigma = \left(I_{p'} \dotplus -I_{q'}\right)$. It is straightforward to verify that $\Sigma^2 = I_n$, $\Sigma R = R\Sigma$, and both R and Σ are symmetric. Then from (27-31) we have $F = R\Sigma F^\top \Sigma R^{-1}$ and $G = -R\Sigma H^\top$. Let V_c and V_o be as in (9,10) with $\mathcal{H}_n$ as in (12). It is straightforward to show that $V_c = -R\Sigma V_o^\top$ and hence

$$\mathcal{H}_n = V_o\left(-R\Sigma\right) V_o^\top.$$

From [15, Theorem 2], the number of positive and negative eigenvalues of $\mathcal{H}_n$ cannot exceed the corresponding quantities for $-R\Sigma$. Since $-R\Sigma = \left(-R_{11} \dotplus R_{22}\right)$ with $-R_{11} < 0$ and $R_{22} > 0$, it follows that $-R\Sigma$ has exactly q' positive and p' negative eigenvalues. From the dimension of the state vector it follows that the McMillan degree of $Z(s)$ is no greater than $n = p' + q'$. Hence, for $\mathcal{H}_k$ as in (16), we have $\pi\left(\mathcal{H}_n\right) = \pi\left(\mathcal{H}_k\right)$ and $\nu\left(\mathcal{H}_n\right) = \nu\left(\mathcal{H}_k\right)$ for all $k \geq \delta\left(Z(s)\right)$, and so $\pi\left(\mathcal{H}_k\right) \leq q' \leq q$ and $\nu\left(\mathcal{H}_k\right) \leq p' \leq p$ for all $k \geq \delta\left(Z(s)\right)$.

Using the argument in Section 5 about the existence of either a proper impedance or a proper admittance we obtain the following theorem which holds irrespective of whether the network is minimally reactive or whether $a(s)$ and $b(s)$ are coprime.

Theorem 10. *Let $Z(s) \in \mathbb{R}(s)$ be as in (19). If $Z(s)$ is the impedance of a reciprocal network containing exactly p inductors and q capacitors then*

$$q \geq \frac{1}{2}(\delta(Z(s)) + \gamma(Z(s))) = \pi(\mathcal{B}(b,a)),$$
$$p \geq \frac{1}{2}(\delta(Z(s)) - \gamma(Z(s))) = \nu(\mathcal{B}(b,a)).$$

Here $\pi(\mathcal{B}(b,a))$ and $\nu(\mathcal{B}(b,a))$ can be calculated in accordance with Theorem 9 providing we replace n with $r = \delta(a(s)/b(s))$.

9 Multi-port networks, generalised Bezoutians, and the extended matrix Cauchy index

The results in this paper generalise in a natural way to multi-port networks. In contrast to the one-port case there is no guarantee of existence of a proper impedance or a proper admittance function. However from [2] any reciprocal m-port network N possesses a scattering matrix description $S(s)$ where

$$\begin{bmatrix} v_1 - i_1 \\ v_2 - i_2 \\ \vdots \\ v_m - i_m \end{bmatrix} = S(s) \begin{bmatrix} v_1 + i_1 \\ v_2 + i_2 \\ \vdots \\ v_m + i_m \end{bmatrix}, \tag{32}$$

and $i_1, v_1, \ldots$ are the Laplace-transformed currents and voltages at the m ports. It is well known that $S(s) \in \mathbb{R}_p^{m\times m}(s)$ and is symmetric [20, Section 2].
Consider the transformation

$$\phi(s) = \frac{s+\alpha}{s-\alpha}, \qquad \alpha > 0, \tag{33}$$

for which

$$\phi^{-1}(s) = \frac{\alpha(s+1)}{s-1},$$

which maps the left half of the s-plane onto the interior of the unit circle in the ϕ-plane. Let

$$\hat{S}(s) = S(\phi^{-1}(s)).$$

It follows from [20, Section 3] that $\hat{S}(s) \in \mathbb{R}_p^{m\times m}(s)$ is symmetric and has a realisation $\hat{S}(s) = J + H(sI - F)^{-1}G$ satisfying $J = J^\top$, $\Sigma F = F^\top \Sigma$, and $\Sigma G = H^\top$ where $\Sigma = (I_p \dotplus -I_q)$ with p (respectively q) the number of inductors (respectively capacitors) in N. It may then be shown that $V_c = \Sigma V_o^\top$ where V_c, V_o are as in (9,10) for $n = p + q \geq \delta(\hat{S}(s))$.
Consider now the infinite Hankel matrix for $\hat{S}(s)$

$$\mathcal{H} = \begin{bmatrix} W_0 & W_1 & W_2 & \cdots \\ W_1 & W_2 & W_3 & \cdots \\ W_2 & W_3 & W_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{34}$$

together with the finite Hankel matrices

$$\mathcal{H}_k = \begin{bmatrix} W_0 & W_1 & \cdots & W_{k-1} \\ W_1 & W_2 & \cdots & W_k \\ \vdots & \vdots & \ddots & \vdots \\ W_{k-1} & W_k & \cdots & W_{2k-2} \end{bmatrix},$$

for $k = 1, 2, \ldots$ where $W_i = HF^iG$ for $i = 0, 1, 2\ldots$ which coincide with the matrices in the Laurent series expansion of $\hat{S}(s)$

$$\hat{S}(s) = W_{-1} + \frac{W_0}{s} + \frac{W_1}{s^2} + \frac{W_2}{s^3} + \ldots. \tag{35}$$

Then from [20, Appendix 1], $r(\mathcal{H}) = r(\mathcal{H}_k) = \delta(\hat{S}(s))$ for all $k \geq \delta(\hat{S}(s))$ (and indeed for all $k \geq r$ where $r \leq \delta(\hat{S}(s))$ is the degree of the least common multiple of all denominators of $\hat{S}(s)$). Furthermore if $\hat{S}(s)$ is symmetric then so too is $\mathcal{H}$ and, as shown in [4], for $k \geq \delta(\hat{S}(s))$ we also have $\sigma(\mathcal{H}) = \sigma(\mathcal{H}_k)$. Since $\mathcal{H}_n = V_oV_c = V_o\Sigma V_o^\top$ and $n \geq \delta(\hat{S}(s))$ then from [15, Theorem 2] (upon a suitable bordering of the matrices $\mathcal{H}_n$ and V_o to make them square and compatible) we have the following.

Theorem 11. *Let $S(s)$ be the scattering matrix of a reciprocal m-port network containing exactly p inductors and q capacitors. Further let $\hat{S}(s) = S(\phi^{-1}(s))$ for $\phi(s)$ as in (33). Then $\hat{S}(s) \in \mathbb{R}_p^{m\times m}(s)$ is symmetric and, with $\mathcal{H}$ as in (34) for $\hat{S}(s)$ written as in (35), we have $p \geq \pi(\mathcal{H})$ and $q \geq \nu(\mathcal{H})$.*

For $\mathcal{H}$ as in (34) with $\hat{S}(s) \in \mathbb{R}_p^{m\times m}(s)$ symmetric and written as in (35), $\sigma(\mathcal{H})$ is equal to the matrix Cauchy index of $\hat{S}(s)$ [4]. To extend these results to the case of non-proper rational matrix functions we introduce the following generalisation of the extended Cauchy index.

Definition 12. For a symmetric matrix $F(s) \in \mathbb{R}^{m\times m}(s)$ we define the extended matrix Cauchy index $\gamma(F(s))$ to be the difference between the number of jumps in the eigenvalues of $F(s)$ from $-\infty$ to $+\infty$ less the number of jumps in the eigenvalues of $F(s)$ from $+\infty$ to $-\infty$ as s increases from a point a through $+\infty$ and then from $-\infty$ to a again, for any $a \in \mathbb{R}$ which is not a pole of $F(s)$.

We remark that $\gamma(F(s))$ is well defined since the eigenvalues of $F(s)$ are defined by algebraic functions [5], and since $F(s)$ has real eigenvalues for any real s, the local power series defining them will not possess fractional powers, hence we can define an extended Cauchy index for each eigenvalue individually and then take the sum.

Definition 12 coincides with the extended Cauchy index of Definition 5 in the scalar case. Furthermore, if $F(s) \in \mathbb{R}_p^{m\times m}(s)$ then $\gamma(F(s))$ coincides with the matrix Cauchy index defined in [4]. Using results in [4] it is straightforward to show the following generalisation of Lemma 6.

Lemma 13. *Let $F(s), F_1(s), F_2(s) \in \mathbb{R}^{m\times m}(s)$ be symmetric. Then*

1. $\gamma(F(s)) = -\gamma(F^{-1}(s))$ *when* $F^{-1}(s)$ *exists.*

2. *If* $F(s) = F_1(s) + F_2(s)$ *and* $\delta(F(s)) = \delta(F_1(s)) + \delta(F_2(s))$ *then* $\gamma(F(s)) = \gamma(F_1(s)) + \gamma(F_2(s))$.

Similar to the scalar case there is a correspondence between the matrix extended Cauchy index and a matrix Bezoutian. If $F(s)$ is a symmetric matrix with a left matrix factorisation $F(s) = B^{-1}(s)A(s)$ ($A(s)$ and $B(s)$ need not be left coprime) then, consistently with [4], we define the matrix Bezoutian $\mathcal{B}(B,A)$ as the symmetric matrix with block entries $\mathcal{B}_{ij}$ satisfying

$$B(z)A^\top(w) - A(z)B^\top(w) = \sum_{i=1}^{n}\sum_{i=1}^{n}\mathcal{B}_{ij}z^{i-1}(z-w)w^{j-1}.$$

This definition coincides with the definition in Section 6 in the scalar case. If $F(s) \in \mathbb{R}_p^{m\times m}(s)$ is symmetric and with left matrix factorisation $F(s) = B^{-1}(s)A(s)$ then, from [1] we have

$$\delta(F(s)) = r(\mathcal{B}(B,A)),$$

and from [4] we obtain

$$\gamma(F(s)) = \sigma(\mathcal{B}(B,A)).$$

We remark that these properties hold irrespective of whether $B(s)$ and $A(s)$ are left coprime. If $F(s)$ is not proper then consider the transformation $\phi(s)$ in (33) for any α which is not a pole of $F(s)$. Then the function $\hat{F}(s) = F\left(\phi^{-1}(s)\right) \in \mathbb{R}_p^{m\times m}(s)$ and we have $\delta\left(\hat{F}(s)\right) = \delta(F(s))$. Since $\phi(s)$ is a monotonically decreasing function of s except at $s = \alpha$, and $\phi(s)$ is rational and bounded at $s = \infty$, it follows that $\gamma\left(\hat{F}(s)\right) = -\gamma(F(s))$. Suppose in addition that $F(s)$ has a left matrix factorisation $F(s) = B^{-1}(s)A(s)$ and let n be the maximum of the degrees of the entries in the matrices $A(s)$ and $B(s)$. It follows that $\hat{F}(s)$ has a left matrix factorisation $\hat{F}(s) = \hat{B}^{-1}(s)\hat{A}(s)$ where

$$\begin{aligned}&\hat{B}(z)\hat{A}^\top(w) - \hat{A}(z)\hat{B}^\top(w)\\ &\quad = (z-1)^n\left(B\left(\phi^{-1}(z)\right)A^\top\left(\phi^{-1}(w)\right) - A\left(\phi^{-1}(z)\right)B^\top\left(\phi^{-1}(w)\right)\right)(w-1)^n.\end{aligned}$$

Then it is straightforward to verify that

$$\mathbf{z}^\top\mathcal{B}\left(\hat{B},\hat{A}\right)\mathbf{w} = -2\alpha\hat{\mathbf{z}}^\top\mathcal{B}(B,A)\,\hat{\mathbf{w}},$$

for all z, w, where

$$\begin{aligned}\mathbf{z}^\top &= \begin{bmatrix}1, & z, & \cdots, & z^{n-1}\end{bmatrix},\\ \mathbf{w}^\top &= \begin{bmatrix}1, & w, & \cdots, & w^{n-1}\end{bmatrix},\\ \hat{\mathbf{z}}^\top &= (z-1)^{n-1}\begin{bmatrix}1, & \alpha\frac{z+1}{z-1}, & \cdots, & \left(\alpha\frac{z+1}{z-1}\right)^{n-1}\end{bmatrix},\\ \hat{\mathbf{w}}^\top &= (w-1)^{n-1}\begin{bmatrix}1, & \alpha\frac{w+1}{w-1}, & \cdots, & \left(\alpha\frac{w+1}{w-1}\right)^{n-1}\end{bmatrix}.\end{aligned}$$

It may be verified that $\hat{\mathbf{w}} = T_1 T_2 \mathbf{w}$ and $\hat{\mathbf{z}} = T_1 T_2 \mathbf{z}$ for

$$T_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ \alpha & 2^1\alpha & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha^{n-2} & \binom{n-2}{1} 2^1 \alpha^{n-2} & \cdots & 2^{n-2}\alpha^{n-2} & 0 \\ \alpha^{n-1} & \binom{n-1}{1} 2^1 \alpha^{n-1} & \cdots & \binom{n-1}{n-2} 2^{n-2}\alpha^{n-1} & 2^{n-1}\alpha^{n-1} \end{bmatrix},$$

$$T_2 = \begin{bmatrix} (-1)^{n-1} & \binom{n-1}{1}(-1)^{n-2} & \cdots & \binom{n-1}{n-2}(-1) & 1 \\ (-1)^{n-2} & \binom{n-2}{1}(-1)^{n-3} & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ -1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

where $\binom{k}{r} = k!/(r!(k-r)!)$. It follows that

$$\mathcal{B}(\hat{B},\hat{A}) = (T_1 T_2)^{\top} (-2\alpha \mathcal{B}(B,A)) (T_1 T_2),$$

and hence $\gamma(F(s)) = -\gamma(\hat{F}(s)) = -\sigma(\mathcal{B}(\hat{B},\hat{A})) = \sigma(\mathcal{B}(B,A))$ and $\delta(F(s)) = \delta(\hat{F}(s)) = r(\mathcal{B}(\hat{B},\hat{A})) = r(\mathcal{B}(B,A))$. We have shown the following.

Lemma 14. *Let $F(s) \in \mathbb{R}^{m\times m}(s)$ be symmetric with left matrix factorisation $F(s) = B^{-1}(s)A(s)$. Then*

$$\delta(F(s)) = r(\mathcal{B}(B,A)),$$
$$\gamma(F(s)) = \sigma(\mathcal{B}(B,A)).$$

We conclude by considering the case when a hybrid matrix description of the behaviour of N is available. By rearranging equation (32) we find

$$(I - \Sigma_e S(s)) \begin{bmatrix} \mathbf{v}_\alpha \\ \mathbf{i}_\beta \end{bmatrix} = (I + \Sigma_e S(s)) \begin{bmatrix} \mathbf{i}_\alpha \\ \mathbf{v}_\beta \end{bmatrix},$$

where $\mathbf{i}_\alpha$, $\mathbf{v}_\alpha$ are the Laplace-transformed vectors of current and voltage across the first m_1 ports, $\mathbf{i}_\beta$, $\mathbf{v}_\beta$ are the Laplace-transformed vectors of current and voltage across the remaining m_2 ports, and $\Sigma_e = (I_{m_1} \dotplus -I_{m_2})$. Hence providing the pertinent inverse exists we have

$$\begin{bmatrix} \mathbf{v}_\alpha \\ \mathbf{i}_\beta \end{bmatrix} = M(s) \begin{bmatrix} \mathbf{i}_\alpha \\ \mathbf{v}_\beta \end{bmatrix}, \tag{36}$$

where

$$M(s)\Sigma_e = -\Sigma_e + 2(\Sigma_e - S(s))^{-1},$$

which is symmetric. Such a Σ_e is commonly referred to as an *external signature matrix*, e.g. [19]. From the properties of the McMillan degree [3, Section 3.6] we have

$$\delta(M(s)\Sigma_e) = \delta(S(s)) = \delta(\hat{S}(s)),$$

and from Lemma 13 and the previous discussion it is straightforward to verify that

$$\gamma(M(s)\Sigma_e) = \gamma(S(s)) = -\gamma(\hat{S}(s)).$$

Combining this with Lemma 14 and Theorem 11 we obtain the following theorem which holds irrespective of whether the network is minimally reactive or whether $A(s)$ and $B(s)$ are left coprime.

Theorem 15. *Let $M(s)$ be the hybrid matrix of an m-port reciprocal network containing exactly p inductors and q capacitors, with current excitation at the first m_1 ports and voltage excitation at the remaining m_2 ports as in (36), and let $\Sigma_e = \left(I_{m_1} \dotplus -I_{m_2}\right)$. Then $M(s)\Sigma_e \in \mathbb{R}^{m\times m}(s)$ is symmetric and, with $M(s)\Sigma_e$ written as a left matrix factorisation $M(s)\Sigma_e = B^{-1}(s)A(s)$, we have*

$$q \geq \frac{1}{2}\left(\delta\left(M(s)\Sigma_e\right) + \gamma(M(s)\Sigma_e)\right) = \pi\left(\mathcal{B}(B,A)\right),$$
$$p \geq \frac{1}{2}\left(\delta\left(M(s)\Sigma_e\right) - \gamma(M(s)\Sigma_e)\right) = \nu\left(\mathcal{B}(B,A)\right).$$

10 Notes

1. (Networks with only one kind of reactive element). It follows from Theorem 7 that any minimally reactive reciprocal one-port network which contains only one kind of reactive element has an impedance function $Z(s) \in \mathbb{R}(s)$ which satisfies $\gamma(Z(s)) = \pm\delta(Z(s))$. This implies that the poles and zeros of $Z(s)$ are real and interlace each other. This is a well-known property of networks with only one kind of reactive element [18]. It is also well-known that any such impedance function can be realised without the aid of transformers in the Cauer and Foster canonical forms.

2. (Poles and zeros of impedance functions). More generally than in 1. Theorem 7 allows connections to be drawn between pole and zero locations of an impedance function $Z(s)$ and the number of inductors and capacitors in any minimally reactive reciprocal realisation of $Z(s)$. In particular, knowledge of all real axis poles and zeros and their multiplicities (including those at infinity) is sufficient to compute the extended Cauchy Index of a positive-real function.

3. (Mechanical networks). The results in this paper apply equally to mechanical networks comprising springs, dampers, inerters and levers with a direct correspondence being provided by the force-current analogy [17].

4. (Identification). The role of the Cauchy index of a proper rational function, equivalently the signature of the corresponding Hankel matrix, is well known in the subject of identification. In [11] it is shown that the $2n$-dimensional parameter space of a strictly proper rational function is divided into $n+1$ connected regions in which there are no pole-zero cancellations, with each such region being characterised by the Cauchy index, and the disconnected regions being separated by rational functions of lower McMillan degree. The original observation is credited to R.W. Brockett [11].

5. (Balanced model order reduction). The Cauchy index of a proper rational function $F(s) = d + c(sI - A)^{-1}b$ is also equal to the signature of the cross-gramian matrices $W_{co}(T) = \int_0^T e^{At}bce^{At}dt$ for $T \geq 0$, and provides insight into the effects of balanced model order reduction on the structural properties of the function [6].

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Bibliography

[1] B. D. O. Anderson and E. I. Jury. Generalized Bezoutian and Sylvester matrices in multivariable linear control. *IEEE Transactions on Automatic Control*, 21(4):551–556, 1976. Cited p. 224.

[2] B. D. O. Anderson and R. W. Newcomb. Cascade connection for time-invariant n-port networks. *Proceedings of the IEE*, 113(6):970–974, 1966. Cited p. 222.

[3] B. D. O. Anderson and S. Vongpanitlerd. *Network Analysis and Synthesis*. Prentice-Hall, 1973. Cited pp. 212, 213, 216, 219, 220, 221, and 225.

[4] R. R. Bitmead and B. D. O. Anderson. The matrix Cauchy index: Properties and applications. *SIAM Journal on Applied Mathematics*, 33(4):655–672, 1977. Cited pp. 223 and 224.

[5] G. A. Bliss. *Algebraic functions*. AMS, 1933. Cited p. 223.

[6] K. V. Fernando and H. Nicholson. On the Cauchy index of linear systems. *IEEE Transactions on Automatic Control*, 28(2):222–224, 1983. Cited p. 226.

[7] R. M. Foster. Academic and theoretical aspects of circuit theory. *Proceeding of the IRE*, 50:866–871, 1962. Cited p. 219.

[8] P. A. Fuhrmann. *A Polynomial Approach to Linear Algebra*. Springer, second edition, 2012. Cited pp. 217 and 218.

[9] F. R. Gantmacher. *The Theory of Matrices*, volume I. Chelsea, 1980. Cited p. 214.

[10] F. R. Gantmacher. *The Theory of Matrices*, volume II. Chelsea, 1980. Cited pp. 214, 215, and 217.

[11] K. Glover. Some geometrical properties of linear systems with implications in identification. *Proceedings of the 6th IFAC World Congress, Boston*, August 24–30 1975. Cited p. 226.

[12] U. Helmke and P. A. Fuhrmann. Bezoutians. *Linear Algebra Appl.*, 122-124:1039–1097, 1989. Cited p. 218.

[13] J. Z. Jiang and M. C. Smith. Regular positive-real functions and five-element network synthesis for electrical and mechanical networks. *IEEE Transactions on Automatic Control*, 56(6):1275–1290, 2011. Cited p. 219.

[14] R. Kalman. Old and new directions of research in system theory. In J. C. Willems, S. Hara, Y. Ohta, and H. Fujioka, editors, *Perspectives in Mathematical System Theory, Control, and Signal Processing*, volume 398 of *LNCIS*, pages 3–13. Springer, 2010. Cited p. 219.

[15] A. M. Ostrowski. A quantitative formulation of Sylvester's law of inertia. *Proceedings of the National Academy of Sciences of the United States of America*, 45(5):740–744, 1959. Cited pp. 214, 221, and 223.

[16] M. Reichert. Die kanonisch und übertragerfrei realisierbaren Zweipolfunktionen zweiten Grades (Transformerless and canonic realisation of biquadratic immittance functions). *Arch. Elek. Übertragung*, 23:201–208, 1969. Cited p. 219.

[17] M. C. Smith. Synthesis of mechanical networks: The inerter. *IEEE Transactions on Automatic Control*, 47(10):1648–1662, 2002. Cited p. 226.

[18] M. E. V. Valkenburg. *Introduction to Modern Network Synthesis*. John Wiley & Sons, 1960. Cited p. 226.

[19] J. C. Willems. Realization of systems with internal passivity and symmetry constraints. *Journal of the Franklin Institute*, 301(6):605–621, 1976. Cited p. 225.

[20] D. C. Youla and P. Tissi. N-port synthesis via reactance extraction, Part I. *IEEE International Convention Record*, 14(7):183–205, 1966. Cited pp. 211, 212, 219, 220, 222, and 223.