A New Look at an Old Problem: Partial Realization via Compressed Sensing (Work in Progress)

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Happy Birthday Malcolm!

Outline

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1. The Partial Realization Problem
2. Relationship to Compressed Sensing
3. Partial Realization via Nuclear Norm Minimization
4. Numerical Examples

Hankel Matrices: Notation

Suppose $\{f_t\}_{t \geq 1}$ is a sequence of real numbers. The associated infinite and finite Hankel matrices are defined as

$$H_{f,\infty} := \begin{bmatrix} f_1 & f_2 & f_3 & \cdots \\ f_2 & f_3 & f_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

$$H_{f,n} := \begin{bmatrix} f_1 & f_2 & \cdots & f_{n-1} & f_n \\ f_2 & f_3 & \cdots & f_n & f_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_n & f_{n+1} & \cdots & f_{2n-2} & f_{2n-1} \end{bmatrix} \in \mathbb{R}^{n \times n}.$$

An Old Theorem

Theorem

(Kronecker (1881)) Suppose $\{f_t\}_{t \geq 1}$ is an ℓ_1 sequence. Then rank$(H_{f,\infty})$ is finite if and only if the power series

$$\tilde{f}(z) = \sum_{t=1}^{\infty} f_t z^{t-1}$$

defines a rational function of z. If so the rank of $H_{f,\infty}$ is the degree of the rational function.

A Basic Fact

Consider a linear discrete-time SISO system

$$x_{t+1} = Ax_t + Bu_t, y_t = Cx_t,$$

where the pairs (A, B) and (C, A) are controllable and observable respectively. Define the unit pulse response and transfer function of the system as

$$h_t = CA^{t-1}B, t \geq 1, \tilde{h}(z) = \sum_{t=1}^{\infty} h_t z^{t-1}.$$

Then the dimension of A is the degree of $\tilde{h}(z)$, which is in turn the rank of $H_{h,\infty}$.

Partial Realization: Problem Formulation

Original: Given a finite sequence $\{h_t\}_{t=1}^m$, find an *infinite sequence* $\{f_t\}_{t \geq 1}$ such that (i) $f(t) = h(t)$ for $t = 1, \ldots, m$ (denoted as $f_1^m = h_1^m$), and (ii) $\text{rank}(H_{f,\infty})$ is minimized.

Realistic: Given a finite sequence $\{h_t\}_{t=1}^m$, and an integer $n \gg m$, find a *finite sequence* $\{f_t\}_{t=1}^{2n-1}$ such that (i) $f_1^m = h_1^m$, and (ii) $\text{rank}(H_{f,n})$ is minimized.

A Simple Observation

Suppose $\{h_t\}_{t=1}^m$ is a subsequence of an infinite sequence $\{h_t\}_{t \geq 1}$ such that $H_{h,\infty}$ has finite rank, say d. Then for all n, $H_{h,n}$ is a submatrix of $H_{h,\infty}$. Hence $\text{rank}(H_{h,n}) \leq d$ for all n. Therefore, for each integer $n \geq 2m-1$,

$$\left\{ \min_{f \in \mathbb{R}^{2n-1}} \text{rank}(H_{f,n}) \text{ s.t. } f_1^m = h_1^m \right\} \leq d.$$

Outline

Matrix Completion Problem

Problem 1: Suppose $X \in \mathbb{R}^{m \times n}$ is a fixed but unknown matrix of rank $k \ll \min\{n, m\}$. Reconstruct X by measuring just some elements of X.

Problem 2: Given integers n, m, and a subset $S \subseteq [m] \times [n]$,[1] and given real numbers $x_{i,j}, (i,j) \in S$, "fill up" the remaining entries so as to minimize the rank. Precisely

$$\min_{Z \in \mathbb{R}^{m \times n}} \text{rank}(Z) \text{ s.t. } z_{ij} = x_{ij} \ \forall (i,j) \in S.$$

Problem 2 is NP-hard.

[1] Here $[n] = \{1, \ldots, n\}$.

The Nuclear Norm of a Matrix

The **nuclear norm** of a matrix A is the sum of its singular values, and is denoted by $\| \cdot \|_N$ (or $\| \cdot \|_*$ by other authors).

The nuclear norm is the **convex envelope** of the rank function (over the unit sphere in the spectral norm); that is, $\| \cdot \|_N$ is the largest convex function that is always $\leq$ the rank of a matrix, over the set of matrix whose maximum singular value is ≤ 1.

Solution via Nuclear Norm Minimization

Replace rank$(\cdot)$ by $\| \cdot \|_N$ as the objective function.

Modified Problem: Given integers n, m, and a subset $S \subseteq [m] \times [n]$, and real numbers $x_{i,j}, (i,j) \in S$,

$$\min_{Z \in \mathbb{R}^{m \times n}} \|Z\|_N \text{ s.t. } z_{ij} = x_{ij} \ \forall (i,j) \in S.$$

This is a *convex* optimization problem and thus tractable.

Available Results on Matrix Completion (Paraphrase)

If a low rank matrix X is sampled uniformly at random, and the above nuclear norm minimization problem is solved, then under suitable "incoherence" conditions, the error $\|Z^* - X\|_S$ is small, with high probability.[2]

[2] Here $\|\cdot\|_S$ denote the spectral norm, i.e., the largest singular value of a matrix.

Outline

Reprise

Recall the notation: If $f \in \mathbb{R}^{2n-1}$, then

$$H_{f,n} := \begin{bmatrix} f_1 & f_2 & \cdots & f_{n-1} & f_n \\ f_2 & f_3 & \cdots & f_n & f_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_n & f_{n+1} & \cdots & f_{2n-2} & f_{2n-1} \end{bmatrix}$$

is an $n \times n$ Hankel matrix.

Problem Formulation

Modified Partial Realization Problem: Given $\{h_t\}_{t=1}^{m}$, and $n \gg m$,

$$\min_{f \in \mathbb{R}^{2n-1}} \|H_{f,n}\|_N \text{ s.t. } f_1^m = h_1^m.$$

This approach seems to work surprisingly well!

Nonstandard Partial Realization Problem

Why specify only *first* m elements of the unit pulse response? Why not specify *some* m elements?

The above problem formulation still makes sense in this case.

In contrast, Nehari's theorem requires one to specify *consecutive* derivatives of a function.

Outline

Example No. 1

A fourth-order system defined by

$$A = \begin{bmatrix} 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 1.0000 \\ 0.3528 & 0.0490 & 0.2300 & 0.1000 \end{bmatrix},$$

$$B = [\ 0 \quad 0 \quad 0 \quad 1\ ]^\top, C = [\ 1 \quad 3 \quad 2 \quad 0\ ],$$

The system poles are at $0.9, -0.8, \pm 0.7\mathbf{i}$.

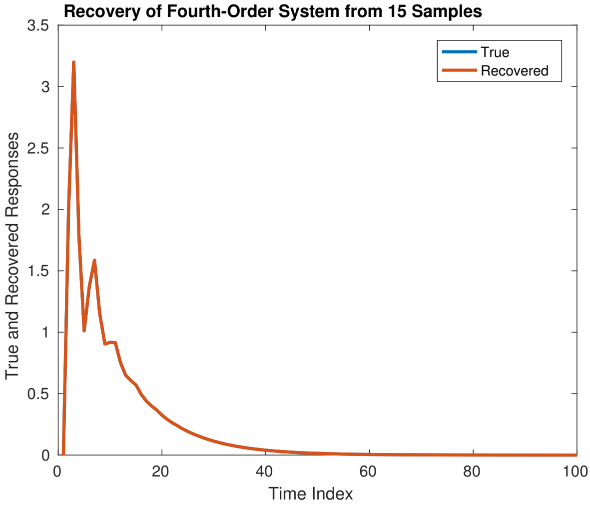
Recovery Using First m Samples

The system is of order 4. Using the first m elements of the unit pulse response, identify the rest using nuclear norm minimization.

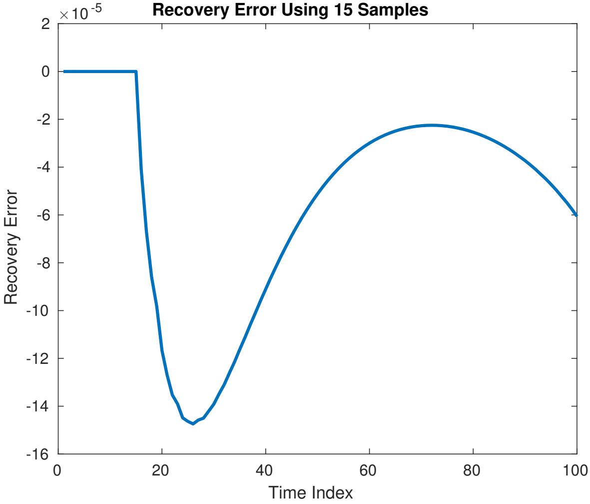
With $n = 50$ (so that $2n - 1 = 99$), and $m = 15$ (so that the first 15 samples are matched), the unit pulse response is recovered.

Results are shown on next few slides.

Recovery Using 15 Samples



Error in Recovery Using 15 Samples



Example No. 2

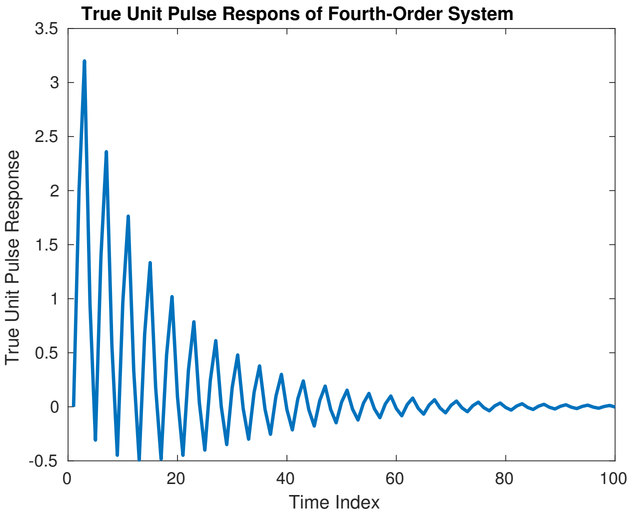
A fourth-order system defined by

$$A = \begin{bmatrix} 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 1.0000 \\ 0.6498 & 0.0902 & -0.1825 & 0.1000 \end{bmatrix},$$

$$B = [\ 0 \quad 0 \quad 0 \quad 1\ ]^\top C = [\ 1 \quad 3 \quad 2 \quad 0\ ].$$

The system poles are at $0.9, -0.8, \pm 0.95\mathbf{i}$, So the system is stable but highly oscillatory, as shown on next slide.

True Unit Pulse Response



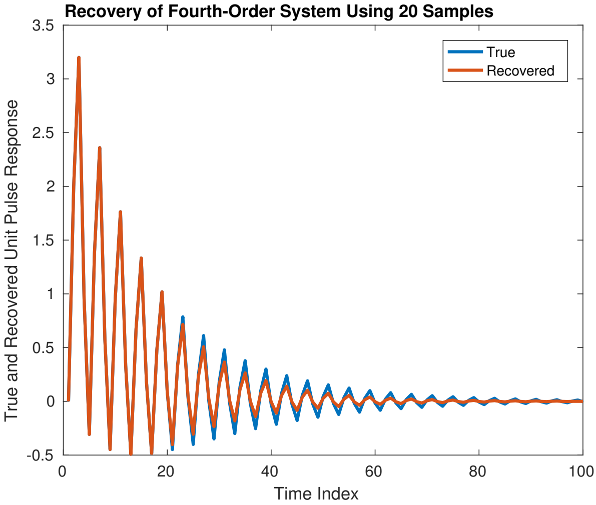
Recovery Using First m Samples

The system is of order 4. Using the first m elements of the unit pulse response, identify the rest using nuclear norm minimization.

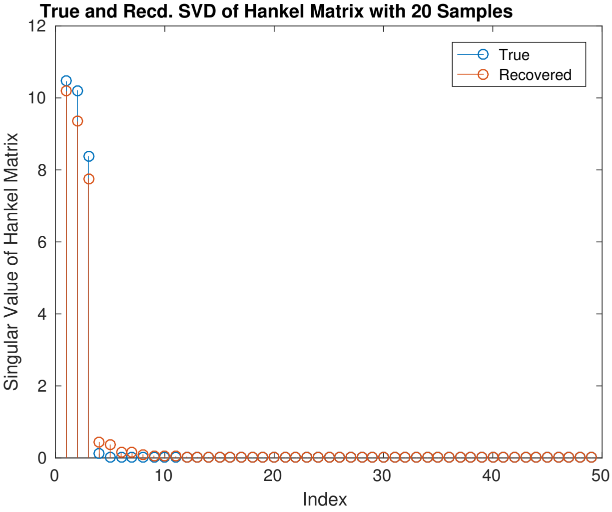
With $n = 50$ (so that $2n - 1 = 99$), and $m = 20$ or 25 (so that the first 20 or 25 samples are matched), the unit pulse response is recovered.

Results are good, and are shown on next few slides.

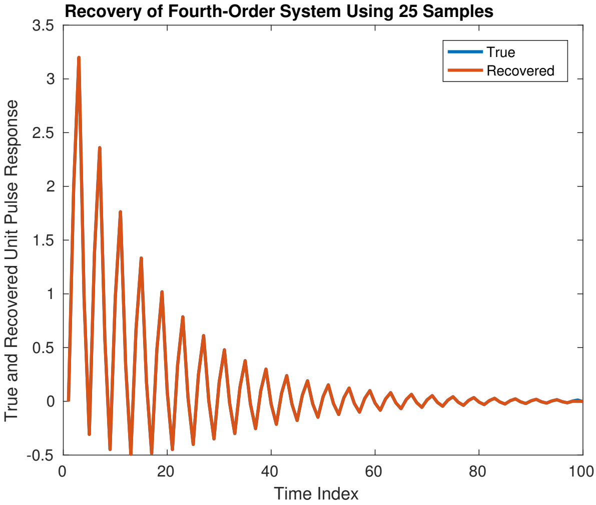
Recovery Using 20 Samples



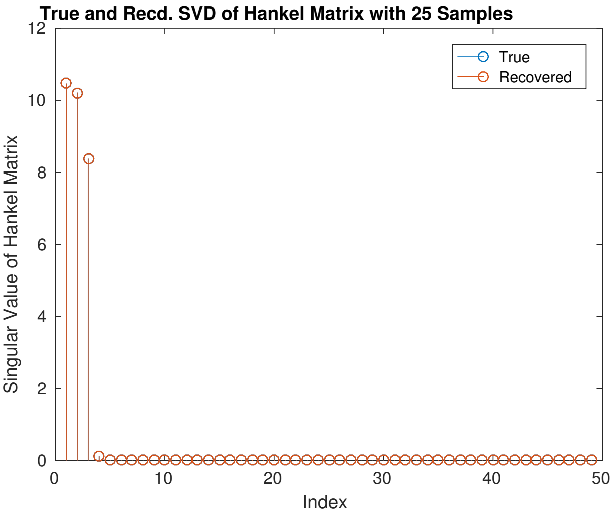
True and Recovered Singular Values of Hankel Matrix



Recovery Using 25 Samples



True and Recovered Singular Values of Hankel Matrix



Partial Realization with Missing Samples

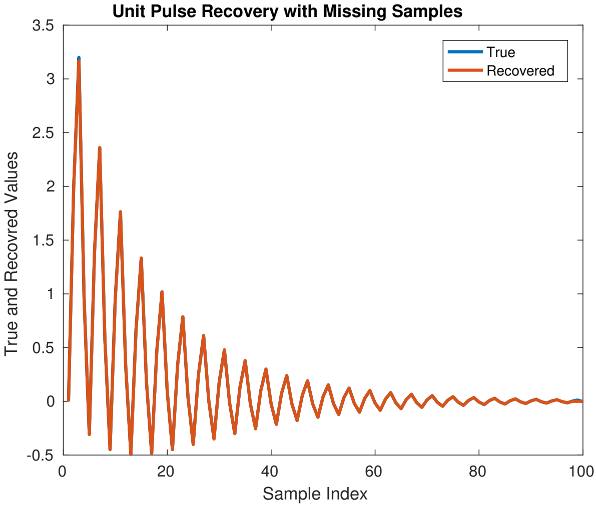
Suppose that out of the first 30 samples, we miss out samples $3, 9, 12, 19, 22$. Define

$$S := \{1, \ldots, 30\} \setminus \{3, 9, 12, 19, 22\}.$$

We minimize the nuclear norm of $H(f)$ subject to the constraint that $f_t = h_t$ for all $t \in S$.

Results shown on next page.

Partial Realization with Missing Samples (Cont'd)



Theoretical Justification?

That is work in progress.

General approach:

- Write down optimality conditions for convex optimization problem (easy!).
- Determine conditions under which *only the true system* satisfies these conditions (hard!)

Questions?